Topological Classification of Certain Wave Equations

in collaboration with Giuseppe De Nittis & Carlos Villegas

Max Lein
Advanced Institute of Materials Research, Tohoku University
2016.06.23@PCS

Overarching Goal

Find the mathematical object whose **topology** determines properties of certain **classical** wave equations.

Making Quantum Analogies Rigorous

Develop and explore the **Schrödinger formalism** for certain *classical wave equations*

- Allows for adaptation of techniques from quantum mechanics to other wave equations
- Also differences, e. g. classical waves $\mathbb{R}$-valued

Some Relevant Wave Equations

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \rho_0 \\ -\rho_0^{-1} \nabla \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^n$

Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

Some Relevant Wave Equations

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \rho_0 \\ -\rho_0^{-1} \nabla \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^n$

Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

Some Relevant Wave Equations

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \rho_0 \\ -\rho_0^{-1} \nabla \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^n$

Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

Some Relevant Wave Equations

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \, \rho_0 \\ -\rho_0^{-1} \, \nabla \, \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^{\dagger}(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^{\dagger}(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^n$

Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

Some Relevant Wave Equations

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \, \rho_0 \\ -\rho_0^{-1} \nabla \, \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^{\dagger}(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^{\dagger}(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^n$

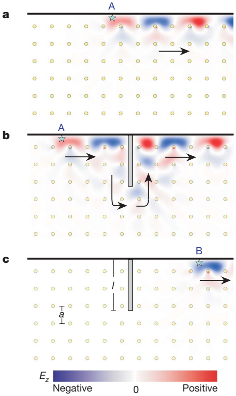
Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

Realizations of *Quantum-Wave Analogies*

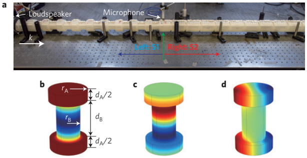
Topological Boundary States

Photonic



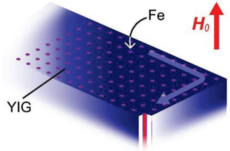
Joannopoulos, Soljačić et al (2009)

Acoustic



Xiao, Ma et al (2015)

Magnonic*



Shindou, Matsumoto et al (2013)

* Not yet realized in experiment.

Despite Experiments ...

... **first-principle derivations** are scarce,
be it rigorous or non-rigorous!

$\rightsquigarrow$ Open field with lots of interesting problems!

Main Messages for Today

1. Recast wave equations in the form of a **Schrödinger equation**.
2. **Classify topological photonic crystals** by their symmetries.
   $\Longrightarrow$ Find mathematical object whose topology is relevant
3. Some media **do *not* fit into the existing Altland-Zirnbauer scheme** (e. g. dual symmetric materials).
4. *Outlook:* Krein-Schrödinger formalism for negative index materials. At present **no classification theory exists**.

1. Schrödinger Formalism for Classical Waves
2. Example: Electromagnetism
3. Classification of Topological Photonic Crystals
4. Open Problem: Classification of Negative Index Materials
5. Encore: Derivation of Approximate Maxwell Equations

1 Schrödinger Formalism for Classical Waves

2 Example: Electromagnetism

3 Classification of Topological Photonic Crystals

4 Open Problem: Classification of Negative Index Materials

5 Encore: Derivation of Approximate Maxwell Equations

Recap: Schrödinger Equation on $\mathbb{R}^d$

Fundamental Constituents

1. Hamilton/Schrödinger operator H, typical examples are

$$H = \tfrac{1}{2m}\big(-\mathrm{i}\nabla - A\big)^2 + V$$
$$H = m\,\beta + \big(-\mathrm{i}\nabla - A\big)\cdot\alpha + V$$

2. Hilbert space $L^2(\mathbb{R}^d, \mathbb{C}^n)$ where $\langle\phi,\psi\rangle = \int_{\mathbb{R}^d}\mathrm{d}x\,\phi(x)\cdot\psi(x)$
3. Dynamics given by Schrödinger equation

$$\mathrm{i}\partial_t\psi(t) = H\psi(t), \qquad \psi(0) = \phi$$

Recap: Schrödinger Equation on $\mathbb{R}^d$

Fundamental Constituents

1. Hamilton/Schrödinger operator H
2. Hilbert space
3. Schrödinger equation

Properties

- $H = H^*$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}t\,H}\phi$
- $\|\psi(t)\|^2 = \|\phi\|^2$ (conservation of propability)

Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* $M = W\, D$ where
   - $W = W^*$, $0 < c\, \mathrm{id} \leq W \leq C\, \mathrm{id}$ (positive, bounded, bounded inverse)
   - $D = D^*$ (potentially unbounded)
2. Complex (!) *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$ where

$$\langle \phi, \psi \rangle_W = \langle \phi, W^{-1} \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x\, \phi(x) \cdot W^{-1} \psi(x)$$

3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\partial_t \psi(t) = M \psi(t), \qquad \psi(0) = \phi$$

4. *Even particle-hole symmetry* K, i. e. K antiunitary, $K^2 = +\mathrm{id}$ and $K\, M\, K = -M$

Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* $M = W\,D$ where
   - $W = W^*$, $0 < c\,\mathrm{id} \leq W \leq C\,\mathrm{id}$
     (positive, bounded, bounded inverse)
   - $D = D^*$ (potentially unbounded)
2. Complex (!) **weighted** *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$ where
   $$\langle \phi, \psi \rangle_W = \langle \phi, W^{-1} \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \phi(x) \cdot W^{-1} \psi(x)$$
3. Dynamics given by *Schrödinger equation*
   $$\mathrm{i}\partial_t \psi(t) = M \psi(t), \qquad \psi(0) = \phi$$
4. *Even particle-hole symmetry* K, i. e.
   K antiunitary, $K^2 = +\mathrm{id}$ and $K\,M\,K = -M$

Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* $M = W \, D$ with **product structure**
2. Complex (!) **weighted** *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$
3. Dynamics given by *Schrödinger equation*
4. *Even particle-hole symmetry* K

Properties

- $M^{*_W} = M$
- $\psi(t) = \mathrm{e}^{-\mathrm{i} t M} \phi$
- $\left\lVert \psi(t) \right\rVert_W^2 = \lVert \phi \rVert_W^2$ (conserved quantity, e. g. energy)
- $\mathrm{Re}_K \, \mathrm{e}^{-\mathrm{i} t M} = \mathrm{e}^{-\mathrm{i} t M} \, \mathrm{Re}_K$ where $\mathrm{Re}_K = \frac{1}{2} (\mathrm{id} + K)$ (existence of real solutions)

Quantum-Wave Analogies

	Wave Equation	**Quantum Mechanics**
Hilbert space	weighted L^2	L^2
Wave function	$\mathbb{R}$-valued	$\mathbb{C}$-valued
Generator dynamics	Maxwell-type operator $M = W\,D = M^*$	Hamiltonian $H = \hat{p}^2 + V = H^*$
Necessary symmetry	+PH	none
Conserved quantity $\lVert\Psi\rVert^2$	e. g. field energy	probability

1 Schrödinger Formalism for Classical Waves

2 Example: Electromagnetism

3 Classification of Topological Photonic Crystals

4 Open Problem: Classification of Negative Index Materials

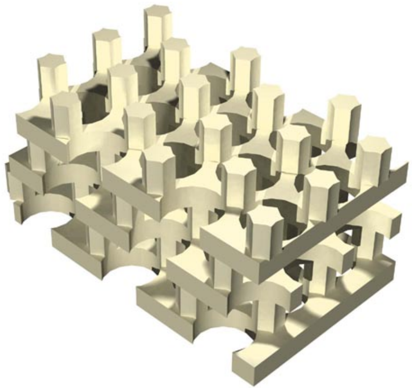
5 Encore: Derivation of Approximate Maxwell Equations

Aim of this Section

Make a first-principles derivation of the Schrödinger formalism for electromagnetic waves, i. e. identify

1. "Hamilton" operator $M = W\, D$
2. Hilbert space
3. Schrödinger equation
4. Even particle-hole symmetry

Maxwell's Equations for Non-Gyrotropic Dielectrics



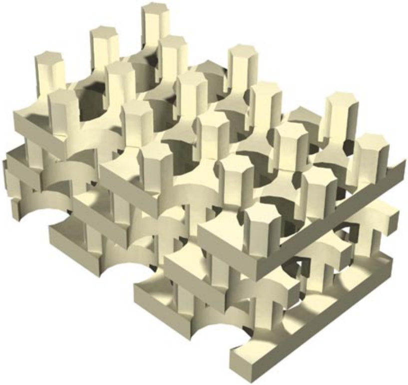
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

1. $W = \overline{W}$ real
   (non-gyrotropic)
2. $W^* = W$ (lossless)
3. $0 < c\,\mathbf{1} \leq W \leq C\,\mathbf{1}$
   (excludes metamaterials)
4. W frequency-independent
   (response instantaneous)

Maxwell's Equations for Non-Gyrotropic Dielectrics



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Schrödinger Formalism of Electromagnetism

$$\left.\begin{array}{c}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}-\nabla\times\mathbf{H}\\ +\nabla\times\mathbf{E}\end{pmatrix}\\ \text{dynamical Maxwell equations}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} \mathrm{i}\partial_t\Psi = M\Psi \\ \text{“Schrödinger-type equation”}\end{array}\right.$$

$$\Psi(t) = \big(\mathbf{E}(t), \mathbf{H}(t)\big) \in \mathcal{H} = \Big\{\Psi \in L^2_W(\mathbb{R}^3, \mathbb{C}^6) \;\big|\; \Psi \text{ transversal}\Big\}$$

$$M = \underbrace{\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}^{-1}}_{=W^{-1}} \underbrace{\begin{pmatrix}0 & +(-\mathrm{i}\nabla)^\times\\ -(-\mathrm{i}\nabla)^\times & 0\end{pmatrix}}_{=\mathrm{Rot}} = M^*$$

$$\left.\begin{array}{c}\text{Maxwell equations}\\ \Longleftrightarrow\\ \text{Maxwell operator } M = M^*\end{array}\right\} \Longrightarrow$$

Adaptation of **techniques from quantum mechanics** to electromagnetism

Fundamental Symmetries of Non-Gyrotropic Materials

$$\left.\begin{matrix}\begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix}\mathbf{E} \\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}-\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E}\end{pmatrix} \\ \text{dynamical Maxwell equations}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\mathrm{i}\partial_t \Psi = M \Psi \\ \text{“Schrödinger-type equation”}\end{matrix}\right.$$

$$W(x) = \begin{pmatrix}\varepsilon(x) & 0 \\ 0 & \mu(x)\end{pmatrix} = \begin{pmatrix}\overline{\varepsilon(x)} & 0 \\ 0 & \overline{\mu(x)}\end{pmatrix} = \overline{W(x)}$$

3 Symmetries

1. $C : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, \overline{\mathbf{H}})$ with $C\,M\,C = -M$ (+PH)
2. $J : (\mathbf{E}, \mathbf{H}) \mapsto (\mathbf{E}, -\mathbf{H})$ with $J\,M\,J = -M$ (χ)
3. $T = J\,C : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, -\overline{\mathbf{H}})$ with $T\,M\,T = +M$ (+TR)

Restriction to Real Fields

$C\,M\,C = -M$ implies

$$\mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0,\mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}\,\Psi_{\pm} = \mathrm{Re}\;\mathrm{e}^{-\mathrm{i}t\,M}\Psi_{\pm}$$

where $\mathrm{Re} \, = \frac{1}{2}(\mathrm{id}+C)$ is the real part operator and

$$\begin{aligned}\Psi_{+} &= 1_{\{\omega>0\}}(M)\,(\mathbf{E}_0,\mathbf{H}_0) = P_{+}(\mathbf{E}_0,\mathbf{H}_0)\\ \Psi_{-} &= 1_{\{\omega<0\}}(M)\,(\mathbf{E}_0,\mathbf{H}_0) = P_{-}(\mathbf{E}_0,\mathbf{H}_0) = C\Psi_{+}\end{aligned}$$

the positive and negative frequency contributions

Restriction to Real Fields

$C\,M\,C = -M$ implies

$$\mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0,\mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}\,\Psi_{\pm} = \mathrm{Re}\ \mathrm{e}^{-\mathrm{i}t\,M}\Psi_{\pm}$$

where $\mathrm{Re} = \frac{1}{2}(\mathrm{id} + C)$ is the real part operator and

$$\Psi_+ = 1_{\{\omega>0\}}(M)\,(\mathbf{E}_0,\mathbf{H}_0) = P_+(\mathbf{E}_0,\mathbf{H}_0)$$
$$\Psi_- = 1_{\{\omega<0\}}(M)\,(\mathbf{E}_0,\mathbf{H}_0) = P_-(\mathbf{E}_0,\mathbf{H}_0) = C\Psi_+$$

the positive and negative frequency contributions

What About Gyrotropic Media?

What if

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \neq \begin{pmatrix} \overline{\varepsilon(x)} & 0 \\ 0 & \overline{\mu(x)} \end{pmatrix} = \overline{W(x)}$$

is complex?

1. Use non-gyrotropic equations $\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$
   $\rightsquigarrow$ often implicitly use in literature, but $\mathrm{Im}\,(\mathbf{E}(t), \mathbf{H}(t)) \neq 0$ ⚡
2. Use $(\mathbf{E}, \mathbf{H}) = \frac{1}{2}(\Psi_+ + \Psi_-)$ and let positive/negative frequency contributions evolve separately via $M = M_+ \oplus M_-$

What About Gyrotropic Media?

What if

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \neq \begin{pmatrix} \overline{\varepsilon(x)} & 0 \\ 0 & \overline{\mu(x)} \end{pmatrix} = \overline{W(x)}$$

is complex?

1. Use non-gyrotropic equations $\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$
   ⇝ often implicitly use in literature, but $\mathrm{Im}\,\big(\mathbf{E}(t), \mathbf{H}(t)\big) \neq 0$ ↯
2. Use $(\mathbf{E}, \mathbf{H}) = \frac{1}{2}(\Psi_+ + \Psi_-)$ and let positive/negative frequency contributions evolve separately via $M = M_+ \oplus M_-$

The Schrödinger Formalism for Gyrotropic Media

$\Psi_{-}(t) = C\Psi_{+}(t)$ can be enforced by choosing $W_{-} = \overline{W_{+}}$, i. e.

$$M_{\pm} = -C\,M_{\mp}\,C = W_{\pm}\,\mathrm{Rot}\big|_{\pm\omega>0}$$

Relation between $M_{\pm}$ implies relation between evolution groups:

$$C\,\mathrm{e}^{-\mathrm{i}M_{\pm}} = \mathrm{e}^{-\mathrm{i}t\,M_{\mp}}\,C$$

The Schrödinger Formalism for Gyrotropic Media

Maxwell equations equivalent to

$$\mathrm{i}\partial_t \Psi(t) = M \Psi(t), \qquad \Psi(0) = \Phi \in \mathcal{H},$$

on the Hilbert space

$$\mathcal{H} := \operatorname{ran} P_+ \oplus \operatorname{ran} P_- \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) \oplus L^2_{W_-}(\mathbb{R}^3, \mathbb{C}^6)$$

with Maxwell operator

$$\begin{aligned} M &:= M_+ \oplus M_- \\ \mathcal{D}(M) &:= \big(P_+ \mathcal{D}(\mathrm{Rot})\big) \oplus \big(P_- \mathcal{D}(\mathrm{Rot})\big) \end{aligned}$$

“Indestructible” Symmetry

$$\left.\begin{matrix} C\, M_+ \, C = -M_- \\ M = M_+ \oplus M_- \end{matrix}\right\} \implies K\, M\, K = -M$$

has an **even particle-hole-type symmetry**

$$K := \sigma_1 \otimes C, \quad (\Psi_+, \Psi_-) \mapsto \big(\overline{\Psi_-}, \overline{\Psi_+}\big).$$

Fundamental Constituents

Complexified Maxwell Equations

1. *"Hamilton" operator* $M = \bigl(W_+ \, \mathrm{Rot} \, \vert_{\mathrm{ran} \, P_+} \bigr) \oplus \bigl(W_- \, \mathrm{Rot} \, \vert_{\mathrm{ran} \, P_-} \bigr)$
2. *Hilbert space*
   $\mathcal{H} = \mathrm{ran} \, P_+ \oplus \mathrm{ran} \, P_- \subset L^2_{W_+}(\mathbb{R}^d, \mathbb{C}^6) \oplus L^2_{W_-}(\mathbb{R}^d, \mathbb{C}^6)$
3. Dynamics given by *Schrödinger equation*
   $$\mathrm{i} \partial_t \psi(t) = M \psi(t) , \qquad \psi(0) = \bigl(P_+(\mathbf{E}, \mathbf{H}) \, , \, P_-(\mathbf{E}, \mathbf{H}) \bigr)$$
4. *Even particle-hole symmetry*: "Complex conjugation"
   $K = \sigma_1 \otimes C$

Reduction to Complex Fields with $\omega > 0$

Physically only **real states** relevant

$$\mathcal{H}_{\mathbb{R}} := \Big\{ (\Psi_+, \overline{\Psi_+}) \;\big|\; \Psi_+ \in \mathrm{ran}\, P_+ \Big\} \subset \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_-$$

$K \, M \, K = -M$ implies

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathrm{Re}\, \big(\mathrm{e}^{-\mathrm{i}t M_+} \Psi_+\big) \simeq \mathrm{e}^{-\mathrm{i}t M} \, \mathrm{Re}_K \, (\Psi_+, 0)$$

where $\mathrm{Re}_K = \frac{1}{2}(\mathrm{id} + K)$ is the real part operator

$$\left. \begin{matrix} \text{Real transversal states} \\ (\mathbf{E}, \mathbf{H}) = \mathrm{Re}\, \Psi_+ \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Complex states with } \omega > 0 \\ \Psi_+ = P_+ (\mathbf{E}, \mathbf{H}) \end{matrix} \right.$$

$\mathrm{Re} = P_+^{-1} \Longrightarrow$ Study symmetries of M_+ (regular, $\pm$TR)

Reduction to Complex Fields with $\omega > 0$

Physically only **real states** relevant

$$\mathcal{H}_{\mathbb{R}} := \Bigl\{ (\Psi_+, \overline{\Psi_+}) \; \big| \; \Psi_+ \in \mathrm{ran}\, P_+ \Bigr\} \subset \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_-$$

$K \, M \, K = -M$ implies

$$\bigl(\mathbf{E}(t), \mathbf{H}(t) \bigr) = \mathrm{Re}\, \bigl(\mathrm{e}^{-\mathrm{i} t M_+} \Psi_+ \bigr) \simeq \mathrm{e}^{-\mathrm{i} t M} \, \mathrm{Re}_K \, (\Psi_+, 0)$$

where $\mathrm{Re}_K = \frac{1}{2} (\mathrm{id} + K)$ is the real part operator

$$\left. \begin{matrix} \text{Real transversal states} \\ (\mathbf{E}, \mathbf{H}) = \mathrm{Re}\, \Psi_+ \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Complex states with } \omega > 0 \\ \Psi_+ = P_+ (\mathbf{E}, \mathbf{H}) \end{matrix} \right.$$

$\mathrm{Re} = P_+^{-1} \Longrightarrow$ Study symmetries of M_+ (regular, $\pm$TR)

Reduction to Complex Fields with $\omega > 0$

Physically only **real states** relevant

$$\mathcal{H}_{\mathbb{R}} := \Bigl\{ (\Psi_+, \overline{\Psi_+}) \;\big|\; \Psi_+ \in \mathrm{ran}\, P_+ \Bigr\} \subset \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_-$$

$K\, M\, K = -M$ implies

$$\bigl(\mathbf{E}(t), \mathbf{H}(t)\bigr) = \mathrm{Re}\, \bigl(\mathrm{e}^{-\mathrm{i} t M_+} \Psi_+\bigr) \simeq \mathrm{e}^{-\mathrm{i} t M}\, \mathrm{Re}_K\, (\Psi_+, 0)$$

where $\mathrm{Re}_K = \frac{1}{2}(\mathrm{id} + K)$ is the real part operator

$$\left.\begin{matrix} \text{Real transversal states} \\ (\mathbf{E}, \mathbf{H}) = \mathrm{Re}\, \Psi_+ \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{Complex states with } \omega > 0 \\ \Psi_+ = P_+ (\mathbf{E}, \mathbf{H}) \end{matrix}\right.$$

$\mathrm{Re} = P_+^{-1} \Longrightarrow$ Study symmetries of M_+ (regular, $\pm$TR)

Fundamental Constituents

Reduced Description

1. *"Hamilton" operator* $M_+ = W_+ \, \mathrm{Rot} \big|_{\mathrm{ran}\, P_+}$
2. *Hilbert space* $\mathcal{H}_+ = \mathrm{ran}\, P_+ \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$
3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H})$$

4. *Even particle-hole symmetry*: Implicit in construction

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathrm{Re}\, \Psi_+(t)$$

1 Schrödinger Formalism for Classical Waves

2 Example: Electromagnetism

3 Classification of Topological Photonic Crystals

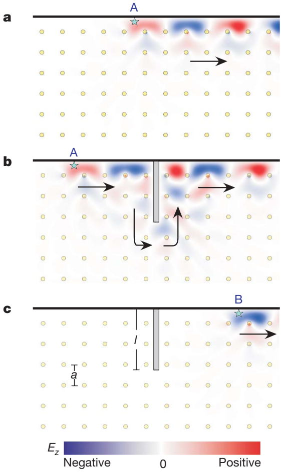
4 Open Problem: Classification of Negative Index Materials

5 Encore: Derivation of Approximate Maxwell Equations

A Novel Class of Materials: *Topological Photonic Crystals*

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \quad \right\} \quad \Longrightarrow$$

symmetry breaking

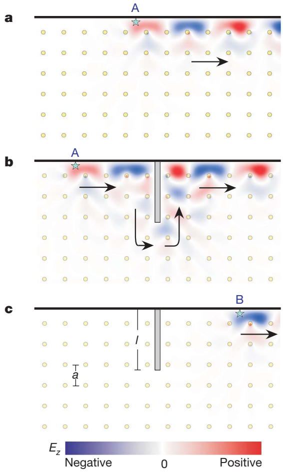


Joannopoulos, Soljačić et al (2009)

A Novel Class of Materials: *Topological Photonic Crystals*

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}\right\} \Longrightarrow$$

symmetry breaking



Joannopoulos, Soljačić et al (2009)

- Photonic bulk-edge correspondences

$\downarrow$

- Identify topological observables
  $O = T +$ error

$\downarrow$

- Find all topological invariants T

$\downarrow$

- Classification of PhCs by symmetries

A Novel Class of Materials: *Topological Photonic Crystals*

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \quad \right\} \Longrightarrow$$

symmetry breaking

- Photonic bulk-edge correspondences

↓

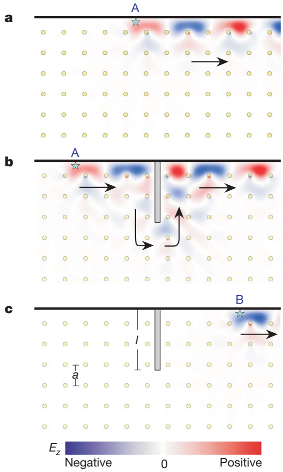
- Identify topological observables
  $O = T + \text{error}$

↓

- Find all topological invariants T

↓

- **Classification of PhCs by symmetries**



Joannopoulos, Soljačić et al (2009)

Which Symmetries *Are* Broken?

Non-Gyrotropic Materials

$$W_+ = \overline{W_+}$$

1 Relevant Symmetry of Complexified Equation

$T : (\psi^E, \psi^H) \mapsto (\overline{\psi^E}, -\overline{\psi^H})$ with $T\, M_+\, T = +M_+$ (+TR)

reverses arrow of time: $T\, \mathrm{e}^{-\mathrm{i} t\, M_+} = \mathrm{e}^{-\mathrm{i} (-t)\, M_+}\, T$

$\Longrightarrow$ *Needs to be broken to have unidirectional edge modes!*

Which Symmetries *Are* Broken?

Gyrotropic Materials

$$W_+ \neq \overline{W_+}$$

1 Relevant Symmetry of Complexified Equation

~~$T : (\psi^E, \psi^H) \mapsto (\overline{\psi^E}, -\overline{\psi^H})$ with $T \, M_+ \, T = +M_+$ (+TR)~~

~~reverses arrow of time: $T \, \mathrm{e}^{-\mathrm{i} t \, M_+} = \mathrm{e}^{-\mathrm{i}(-t) \, M_+} \, T$~~

$\Longrightarrow$ *Needs to be broken to have unidirectional edge modes!*

The Topology of Light States in Periodic Media

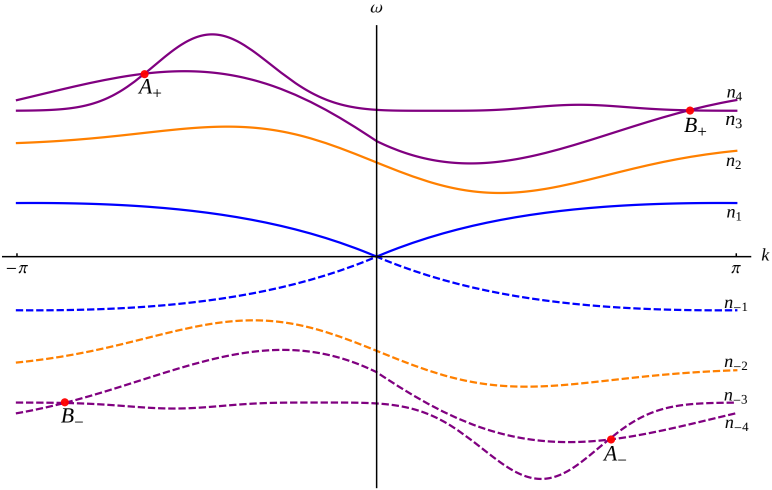
$$\left.\begin{array}{c}\text{Existence of topological}\\ \text{boundary states}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Topology of the (bulk)}\\ \text{Bloch bundle}\\ \mathcal{E}_{\mathrm{Bloch}} = \big(\xi_{\mathrm{Bloch}} \xrightarrow{\pi} \mathbb{T}^3\big)\end{array}\right.$$

where $\xi_{\mathrm{Bloch}} = \bigsqcup_{k \in \mathbb{T}^3} \mathrm{span}\{\varphi_n(k)\}_{n \in \mathcal{J}}$ is associated to **finitely many frequency bands*** separated by a **spectral gap** from the others. $\mathcal{E}_{\mathrm{Bloch}}$ may be endowed with **symmetries**.

What is the correct bundle here?

* Not ground state bands!

The Topology of Light States in Periodic Media



The Topology of Light States in Periodic Media

$$\left.\begin{array}{c}\text{Existence of topological}\\ \text{boundary states}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Topology of the (bulk)}\\ \text{Bloch bundle}\\ \mathcal{E}_{\text{Bloch}} = \left(\xi_{\text{Bloch}} \xrightarrow{\pi} \mathbb{T}^3\right)\end{array}\right.$$

where $\xi_{\text{Bloch}} = \bigsqcup_{k \in \mathbb{T}^3} \mathrm{span}\{\varphi_n(k)\}_{n \in \mathcal{J}}$ is associated to finitely many frequency bands* separated by a spectral gap from the others. $\mathcal{E}_{\text{Bloch}}$ may be endowed with symmetries.

What is the correct bundle here?

* Not ground state bands!

Which Schrödinger Framework to Choose?

$$M = M_+ \oplus M_-$$

Choose bands symmetrically: $\{\omega_n(k), -\omega_n(-k)\}$

$$\mathcal{E}_{\mathrm{Bloch}} = \mathcal{E}_+ \oplus \mathcal{E}_- \cong \mathcal{E}_+ \oplus \mathcal{E}_+^*$$

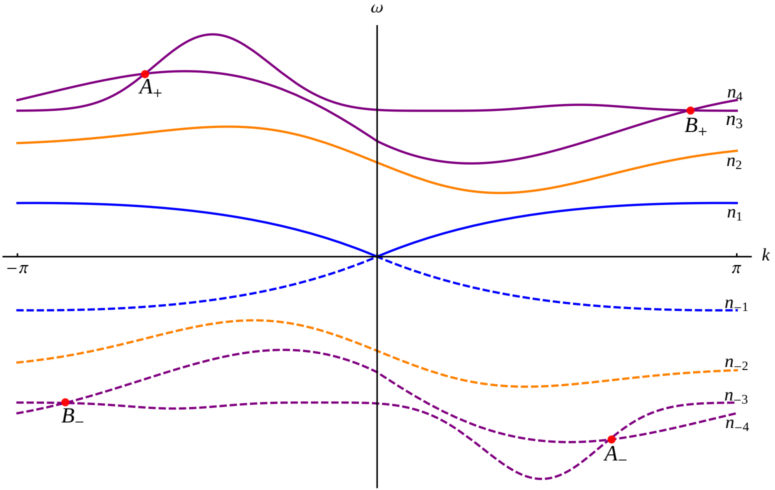
But: $\mathcal{E}_+ \oplus \mathcal{E}_-$ is **always trivial** complex vector bundle as

$$c_1(\mathcal{E}_-) = -c_1(\mathcal{E}_+)$$

$\Longrightarrow$ Unable to predict existence of topological edge modes

Reason: $\mathcal{E}_+ \oplus \mathcal{E}_-$ too big, contains many unphysical states

Which Schrödinger Framework to Choose?



Which Schrödinger Framework to Choose?

$$M = M_+ \oplus M_-$$

Choose bands symmetrically: $\{\omega_n(k), -\omega_n(-k)\}$

$$\mathcal{E}_{\mathrm{Bloch}} = \mathcal{E}_+ \oplus \mathcal{E}_- \cong \mathcal{E}_+ \oplus \mathcal{E}_+^*$$

But: $\mathcal{E}_+ \oplus \mathcal{E}_-$ is **always trivial** complex vector bundle as

$$c_1(\mathcal{E}_-) = -c_1(\mathcal{E}_+)$$

$\Longrightarrow$ Unable to predict existence of topological edge modes
Reason: $\mathcal{E}_+ \oplus \mathcal{E}_-$ too big, contains many unphysical states

Which Schrödinger Framework to Choose?

$$M = M_+ \oplus M_-$$

Choose bands symmetrically: $\{\omega_n(k), -\omega_n(-k)\}$

$$\mathcal{E}_{\text{Bloch}} = \mathcal{E}_+ \oplus \mathcal{E}_- \cong \mathcal{E}_+ \oplus \mathcal{E}_+^*$$

But: $\mathcal{E}_+ \oplus \mathcal{E}_-$ is **always trivial** complex vector bundle as

$$c_1(\mathcal{E}_-) = -c_1(\mathcal{E}_+)$$

$\Longrightarrow$ Unable to predict existence of topological edge modes

Reason: $\mathcal{E}_+ \oplus \mathcal{E}_-$ too big, contains many unphysical states

$$M_+$$

Choose only $\omega_n(k) > 0$

$$\mathcal{E}_{\text{Bloch}} = \mathcal{E}_+$$

$\mathcal{E}_+$ **can be non-trivial**

$$c_1(\mathcal{E}_+) \neq 0$$

Complex sections in $\mathcal{E}_+$ in 1-to-1 correspondence with **real** states

Which Schrödinger Framework to Choose?

$$M = M_+ \oplus M_-$$

Choose bands symmetrically: $\{\omega_n(k), -\omega_n(-k)\}$

$$\mathcal{E}_{\mathrm{Bloch}} = \mathcal{E}_+ \oplus \mathcal{E}_- \cong \mathcal{E}_+ \oplus \mathcal{E}_+^*$$

But: $\mathcal{E}_+ \oplus \mathcal{E}_-$ is **always trivial** complex vector bundle as

$$c_1(\mathcal{E}_-) = -c_1(\mathcal{E}_+)$$

$\Longrightarrow$ Unable to predict existence of topological edge modes

Reason: $\mathcal{E}_+ \oplus \mathcal{E}_-$ too big, contains many unphysical states

$$M_+$$

Choose only $\omega_n(k) > 0$

$$\mathcal{E}_{\mathrm{Bloch}} = \mathcal{E}_+$$

$\mathcal{E}_+$ **can be non-trivial**

$$c_1(\mathcal{E}_+) \neq 0$$

Complex sections in $\mathcal{E}_+$ in 1-to-1 correspondence with **real** states

Comparison Between Photonics and Quantum Mechanics

Theorem (Classification via $\mathcal{E}_+$, De Nittis-L., 2016)

Material	Photonics	Quantum Mechanics
ordinary dielectric	class AI +TR	class AI +TR
exhibiting edge currents	class A none	class A/AII none/-TR
vacuum & dual-symmetric	*New!* 2 anticommuting +TR	
metals (non-rigorous, $W \not> 0$)	*New!* commuting +TR & -TR	

Comparison Between Photonics and Quantum Mechanics

Theorem (Classification via $\mathcal{E}_+$, De Nittis-L., 2016)

Material	Photonics	Quantum Mechanics
ordinary dielectric	class AI +TR	class AI +TR
exhibiting edge currents	class A none	class A/AII none/-TR
vacuum & dual-symmetric	*New!* 2 **anticommuting** +TR	
metals (non-rigorous, $W \not> 0$)	*New!* **commuting** +TR & -TR	

Beyond the Altland-Zirnbauer Classification

Classification of systems

- with *two* time-reversal symmetries
- on **Krein spaces**
- with dispersion
- with non-linearities

Open Questions

1. Identify all topological classes.
2. Find physically relevant cases. *(Pick your battles!)*
3. Find all topological invariants for a given class?

Beyond the Altland-Zirnbauer Classification

Classification of systems

- with *two* time-reversal symmetries
- on **Krein spaces**
- with dispersion
- with non-linearities

Open Questions

1. Identify all topological classes.
2. Find physically relevant cases. *(Pick your battles!)*
3. Find all topological invariants for a given class?

Idea of using *positive frequency bundle* $\mathcal{E}_+$
in classification of topological insulators
should extend to other classical wave equations!

Conclusion

- Schrödinger formalism for classical waves
- $\mathbb{R}$-valuedness of classical waves

$$\left.\begin{matrix}\text{Real transversal states}\\ (\mathbf{E},\mathbf{H}) = \mathrm{Re}\,\Psi_+\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Complex states with } \omega > 0\\ \Psi_+ = P_+(\mathbf{E},\mathbf{H})\end{matrix}\right.$$

$\Longrightarrow$ Particle-hole symmetry irrelevant for topological effects

- Systematic classification of wave equations according to discrete symmetries
- Periodic case:

$$\left.\begin{matrix}\text{Topology}\\ \text{of system}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Topology of}\\ \text{Bloch vector bundle}\\ \text{with symmetries}\end{matrix}\right.$$

- Novel, unexplored topological classes outside of 10-Fold Way

Thank you for your attention!

1 Schrödinger Formalism for Classical Waves

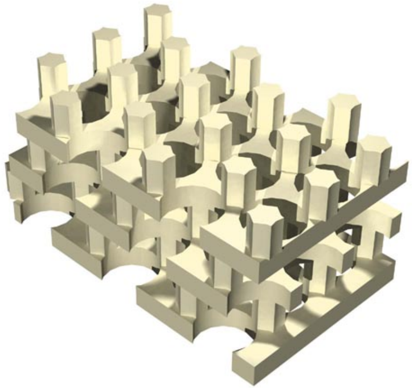
2 Example: Electromagnetism

3 Classification of Topological Photonic Crystals

4 Open Problem: Classification of Negative Index Materials

5 Encore: Derivation of Approximate Maxwell Equations

Simple Model for Negative Index Materials



Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. $W^* = W$ (lossless)
2. $0 < c\,\mathbf{1} \leq |W| \leq C\,\mathbf{1}$ (excludes "zero index" materials)
3. W frequency-independent (response instantaneous)

Simple Model for Negative Index Materials

Theory of Krein spaces

$$\langle \Phi , \Psi \rangle_{W_+} = \langle \Phi , W^{-1} \, \Psi \rangle$$
$$\lVert \Psi \rVert_W = \langle \Psi , W^{-1} \, \Psi \rangle \not\geq 0$$

$\Longrightarrow M = W \, \mathrm{Rot}$ **not** hermitian but ***Krein*-hermitian**

$\leadsto$ Develop Schrödinger-Krein formalism for waves

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. $W^* = W$ (lossless)
2. $0 < c \, \mathbf{1} \leq |W| \leq C \, \mathbf{1}$ (excludes "zero index" materials)
3. W frequency-independent (response instantaneous)

Wigner's Theorem for Krein Spaces

Theorem (Wigner's Theorem)

*Any symmetry operation with on a **Hilbert space** $\mathcal{H}$ is implemented by an operator U with*

$$\langle U\Phi, U\Psi \rangle = \begin{cases} \langle \Phi, \Psi \rangle & U \text{ unitary} \\ \overline{\langle \Phi, \Psi \rangle} & U \text{ antiunitary} \end{cases}$$

Wigner's Theorem for Krein Spaces

Theorem (Wigner's Theorem on Krein spaces [Bracci, Morchio, Strocci (1974)])

*Any symmetry operation with on a **Krein space** $\mathscr{H}$ is implemented by an operator U with*

$$\langle U\Phi, U\Psi\rangle_W = \begin{cases} \pm\langle\Phi,\Psi\rangle_W & U \text{ Krein-(para)-unitary} \\ \pm\overline{\langle\Phi,\Psi\rangle_W} & U \text{ Krein-(para)-antiunitary} \end{cases}$$

Suggests that a **much richer classification theory** (beyond Altland-Zirnbauer) for negative index materials!

Wigner's Theorem for Krein Spaces

Theorem (Wigner's Theorem on Krein spaces [Bracci, Morchio, Strocci (1974)])

*Any symmetry operation with on a **Krein space** $\mathcal{H}$ is implemented by an operator U with*

$$\langle U\Phi, U\Psi\rangle_W = \begin{cases} \pm\langle\Phi,\Psi\rangle_W & U \text{ Krein-(para)-unitary} \\ \pm\overline{\langle\Phi,\Psi\rangle_W} & U \text{ Krein-(para)-antiunitary} \end{cases}$$

Suggests that a **much richer classification theory** (beyond Altland-Zirnbauer) for negative index materials!

Wigner's Theorem for Krein Spaces

Challenges

- Find suitable Helmholtz splitting
- Define $M_{\pm}$: restriction to $\omega > 0$ and $\omega < 0$ tricky
- $M_{\pm}$ and M *no longer hermitian* (selfadjoint) but Krein-hermitian
- *4 additional* (!) types of symmetries

Guinea pig system: *Magnons propagating in a magnonic crystal*
(similar, but not unitarily equivalent
to a hermitian system, explicit)

1 Schrödinger Formalism for Classical Waves

2 Example: Electromagnetism

3 Classification of Topological Photonic Crystals

4 Open Problem: Classification of Negative Index Materials

5 Encore: Derivation of Approximate Maxwell Equations

Fundamental Equations

Maxwell's equations in media

1. *Maxwell's equations*

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}+\nabla\times\mathbf{H}\\ -\nabla\times\mathbf{E}\end{pmatrix} - \begin{pmatrix}J^D\\ J^B\end{pmatrix} \qquad \text{(dynamical eqns.)}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix} = \begin{pmatrix}\rho^D\\ \rho^B\end{pmatrix} \qquad \text{(constraint eqns.)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \mathcal{W}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge*

$$\nabla\cdot J^\sharp + \rho^\sharp = 0, \quad \sharp = D, B$$

Fundamental Equations

Maxwell's equations in media

1. *Maxwell's equations*

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}+\nabla\times\mathbf{H}\\ -\nabla\times\mathbf{E}\end{pmatrix} \qquad \text{(dynamical eqns.)}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \qquad \text{(constraint eqns.)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = W\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge* $\rightsquigarrow$ neglect sources for simplicity

$$\nabla\cdot J^{\sharp} + \rho^{\sharp} = 0, \quad \sharp = D, B$$

Constitutive Relations

For a **linear** medium the constitutive relations maps a **trajectory**

$$(-\infty, t] \ni s \mapsto \big(\mathbf{E}(s), \mathbf{H}(s)\big)$$

onto

$$\begin{pmatrix} \mathbf{D}(t,x) \\ \mathbf{B}(t,x) \end{pmatrix} := \int_{-\infty}^{t} \mathrm{d}s \, W(t-s,x) \begin{pmatrix} \mathbf{E}(s,x) \\ \mathbf{H}(s,x) \end{pmatrix}$$

$\rightsquigarrow$ reaction of medium to impinging em wave depends on the **past**

Constitutive Relations

$$\big(\mathbf{D}(t), \mathbf{B}(t)\big) := \int_{-\infty}^{t} \mathrm{d}s \, W(t-s) \, \big(\mathbf{E}(s), \mathbf{H}(s)\big)$$

Assumption (Constitutive relations)

We assume that $W(t,x) = \begin{pmatrix} \varepsilon(t,x) & \chi^{EH}(t,x) \\ \chi^{HE}(t,x) & \mu(t,x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$

1. *is real,* $W = \overline{W}$*, and*
2. *satisfies the causality condition* $W(t) = 0$ *for all* $t > 0$*.*

Constitutive Relations

$$(\mathbf{D}(t), \mathbf{B}(t)) = (W * (\mathbf{E}, \mathbf{H}))(t)$$

Assumption (Constitutive relations)

We assume that $W(t,x) = \begin{pmatrix} \varepsilon(t,x) & \chi^{EH}(t,x) \\ \chi^{HE}(t,x) & \mu(t,x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$

1. *is real,* $W = \overline{W}$*, and*
2. *satisfies the causality condition* $W(t) = 0$ *for all* $t > 0$*.*

Rewriting the Dynamical Equations

$$\frac{\partial}{\partial t} W * \Psi = -\mathrm{i}\,\mathrm{Rot}\,\Psi := -\mathrm{i} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} \Psi$$

$$\Longleftrightarrow$$

$$\mathrm{i}\,\frac{\partial}{\partial t} W * \Psi = \mathrm{Rot}\,\Psi$$

where $\Psi = (\mathbf{E}, \mathbf{H})$ is the electromagnetic field

Rewriting the Dynamical Equations

$$\mathrm{i}\,\tfrac{\partial}{\partial t} W * \Psi = \mathrm{Rot}\,\Psi$$

$$\big\downarrow \mathcal{F}^{-1}$$

$$\omega\,\widehat{W}(\omega)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

Reality condition implies

$$W = \overline{W} \quad \Longleftrightarrow \quad \widehat{W}(-\omega) = \overline{\widehat{W}(\omega)}$$

Real solutions = linear combinations of complex waves of $\pm\omega(\pm k)$

$$\cos(k \cdot x - \omega t) = \frac{1}{2}\Big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)} + \mathrm{e}^{-\mathrm{i}(k \cdot x - t\omega)}\Big) = \mathrm{Re}\,\big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)}\big)$$

$$\sin(k \cdot x - \omega t) = \frac{1}{\mathrm{i}2}\Big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)} - \mathrm{e}^{-\mathrm{i}(k \cdot x - t\omega)}\Big) = \mathrm{Im}\,\big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)}\big)$$

Rewriting the Dynamical Equations

$$\mathrm{i}\,\frac{\partial}{\partial t} W * \Psi = \mathrm{Rot}\,\Psi$$

$$\Big\downarrow \mathscr{F}^{-1}$$

$$\omega\,\widehat{W}(\omega)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

Reality condition implies

$$W = \overline{W} \quad \Longleftrightarrow \quad \widehat{W}(-\omega) = \overline{\widehat{W}(\omega)}$$

Real solutions = linear combinations of complex waves of $\pm\omega(\pm k)$

$$\cos(k \cdot x - \omega t) = \frac{1}{2}\Big(\mathrm{e}^{+\mathrm{i}((+k)\cdot x - t(+\omega))} + \mathrm{e}^{+\mathrm{i}((-k)\cdot x - t(-\omega))}\Big)$$

$$\sin(k \cdot x - \omega t) = \frac{1}{\mathrm{i}2}\Big(\mathrm{e}^{+\mathrm{i}((+k)\cdot x - t(+\omega))} - \mathrm{e}^{+\mathrm{i}((-k)\cdot x - t(-\omega))}\Big)$$

Approximate Maxwell Equations for $\omega \approx \omega_0$

$$\mathrm{i} \, \frac{\partial}{\partial t} W * \Psi = \mathrm{Rot} \, \Psi$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$\omega \, \widehat{W}(\omega) \, \widehat{\Psi}(\omega) = \mathrm{Rot} \, \widehat{\Psi}(\omega)$$

1. Approximate material weights $\widehat{W}(\omega) \approx \widehat{W}(\omega_0)$
2. Undo Fourier transform

Approximate Maxwell Equations for $\omega \approx \omega_0$

$$\mathrm{i} \, \tfrac{\partial}{\partial t} W * \Psi = \mathrm{Rot} \, \Psi$$

$$\omega \, \widehat{W}(\omega_0) \, \widehat{\Psi}(\omega) = \mathrm{Rot} \, \widehat{\Psi}(\omega)$$

1. Approximate material weights $\widehat{W}(\omega) \approx \widehat{W}(\omega_0)$
2. Undo Fourier transform

Approximate Maxwell Equations for $\omega \approx \omega_0$

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\widehat{W}(\omega_0)\,\Psi = \mathrm{Rot}\,\Psi$$

$$\Big\uparrow\ \mathcal{F}$$

$$\omega\,\widehat{W}(\omega_0)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

1. Approximate material weights $\widehat{W}(\omega) \approx \widehat{W}(\omega_0)$
2. Undo Fourier transform

Approximate Maxwell Equations for $\omega \approx \omega_0$

$$\widehat{W}(\omega_0) \, \mathrm{i} \, \tfrac{\partial}{\partial t} \Psi = \mathrm{Rot} \, \Psi$$

$$\Big\uparrow \mathcal{F}$$

$$\omega \, \widehat{W}(\omega_0) \, \widehat{\Psi}(\omega) = \mathrm{Rot} \, \widehat{\Psi}(\omega)$$

1. Approximate material weights $\widehat{W}(\omega) \approx \widehat{W}(\omega_0)$
2. Undo Fourier transform

Approximate Maxwell Equations for $\omega \approx \omega_0$

Real solutions linear combination of $\pm\omega \rightsquigarrow$ pair of equations

$$\omega > 0 : \qquad \begin{cases} \widehat{W}(\omega_0) \, \mathrm{i}\partial_t \Psi = \mathrm{Rot}\, \Psi \\ \mathrm{Div}\, \widehat{W}(\omega_0) \, \Psi = 0 \end{cases}$$

$$\omega < 0 : \qquad \begin{cases} \widehat{W}(-\omega_0) \, \mathrm{i}\partial_t \Psi = \mathrm{Rot}\, \Psi \\ \mathrm{Div}\, \widehat{W}(-\omega_0) \, \Psi = 0 \end{cases}$$

$$W = \overline{W} \quad \Longleftrightarrow \quad \widehat{W}(-\omega_0) = \overline{\widehat{W}(\omega_0)}$$

Approximate Maxwell Equations for $\omega \approx \omega_0$

Real solutions linear combination of $\pm\omega \rightsquigarrow$ pair of equations

$$\omega > 0 : \qquad \begin{cases} \widehat{W}(\omega_0) \, \mathrm{i}\partial_t \Psi = \mathrm{Rot} \, \Psi \\ \mathrm{Div} \, \widehat{W}(\omega_0) \, \Psi = 0 \end{cases}$$

$$\omega < 0 : \qquad \begin{cases} \overline{\widehat{W}(\omega_0)} \, \mathrm{i}\partial_t \Psi = \mathrm{Rot} \, \Psi \\ \mathrm{Div} \, \overline{\widehat{W}(\omega_0)} \, \Psi = 0 \end{cases}$$

$$W = \overline{W} \quad \Longleftrightarrow \quad \widehat{W}(-\omega_0) = \overline{\widehat{W}(\omega_0)}$$