Classification of Photonic Topological Insulators and Their Effective Dynamics

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in collarboration with Giuseppe De Nittis

AIMR

2015.09.28@Tohoku University

Talk Based on

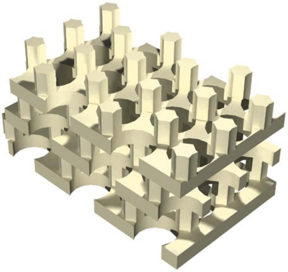
Collaboration with **Giuseppe De Nittis**

- *On the Role of Symmetries in the Theory of Photonic Crystals*
  Annals of Physics **350**, pp. 568--587, 2014
- *Effective Light Dynamics in Perturbed Photonic Crystals*
  Comm. Math. Phys. **332**, issue 1, pp. 221--260, 2014
- *Derivation of Ray Optics Equations in Photonic Crystals Via a Semiclassical Limit*
  arxiv:1502.07235, submitted for publication, 2015

Periodic Light Conductors

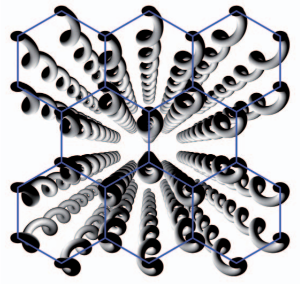
Photonic Crystals

Johnson & Joannopoulos (2004)



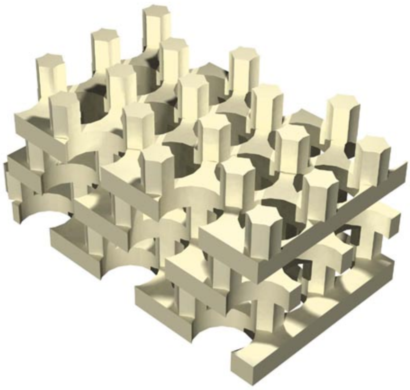
Periodic Waveguide Arrays

Rechtsman, Szameit et al (2013)



- periodic structure $\Longrightarrow$ *peculiar light conduction properties*
- *"band structure engineering"*
  $\rightsquigarrow$ **photonic band gaps**, slow light, low-dispersion materials
- artificial PLCs can be *engineered arbitrarily* and *inexpensively*
- natural photonic crystals: gem stones, beetle shells, butterfly wings

Periodic Light Conductors



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Maxwell equations

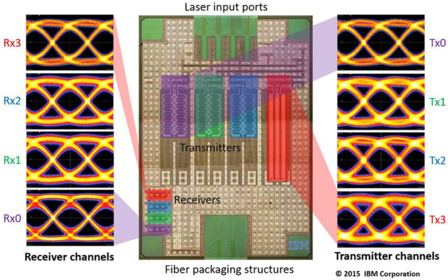
Dynamical equations

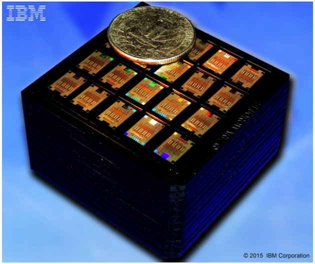
$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \nabla \cdot \big(\varepsilon \mathbf{E} + \chi \mathbf{H} \big) \\ \nabla \cdot \big(\chi^* \mathbf{E} + \mu \mathbf{H} \big) \end{pmatrix} = 0$$

PLCs as *Building Blocks* of Future Photonic Devices





Wavelength-multiplexed silicon photonics chip by IBM (2015)

Integration of photonics and electronics

- Moving photonics closer to the CPUs and GPUs
- Server chassis $\rightarrow$ motherboard $\rightarrow$ chip package $\rightarrow$ die
- Ultimately optical processing
  (eliminates need for optical-electrical-optical conversion)
- *Advantages:* lower power consumption, no interference,
  higher bandwidth, lower latency, longer distances

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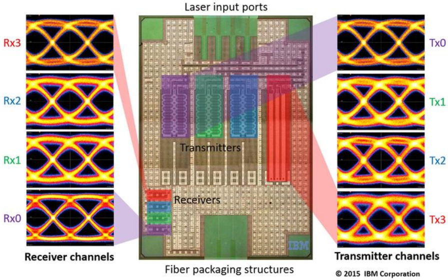

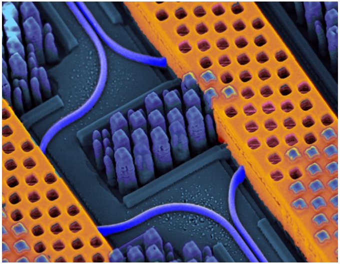

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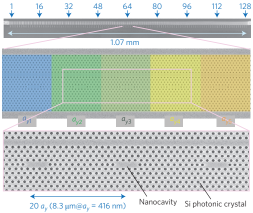
PLCs as *Building Blocks* of Future Photonic Devices

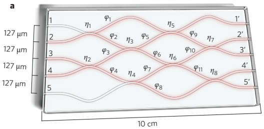
PhC-based photonic memory

Kuramochi et al, NTT Nanophotonics Center (2014)



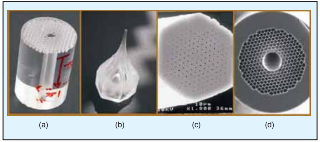
Quantum computing logic

Szameit et al, Jena (2013)



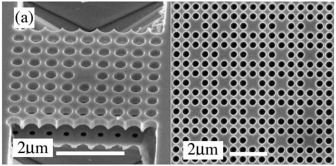
PhC-based waveguides

Russel, Max Planck Institute for Light (2014)



PhC-based laser cavities

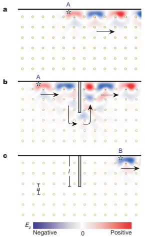
Altug and Vučković, Stanford (2005)



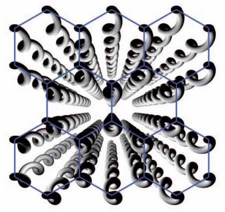
A Novel Class of Materials: Photonic Topological Insulators

$$\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$$

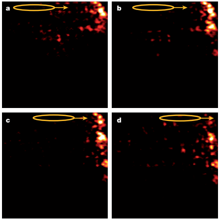
symmetry breaking $\Longrightarrow$



Joannopoulos, Soljačić et al (2009)



constrained,
non-trivial geometry $\Longrightarrow$



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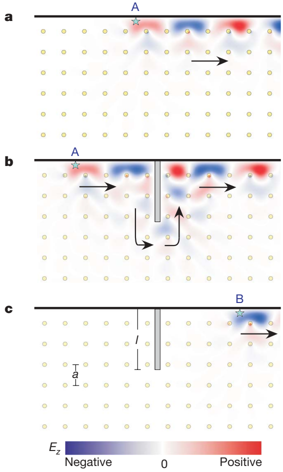
$$\left.\begin{matrix}\begin{pmatrix}\overline{\varepsilon} & 0 \\ 0 & \overline{\mu}\end{pmatrix} \neq \begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix} \\ \text{symmetry breaking}\end{matrix}\right\} \Longrightarrow$$



Joannopoulos, Soljačić et al (2009)

A Novel Class of Materials: Photonic Topological Insulators

Understand ***how topological effects emerge from electrodynamics****,*
starting from Maxwells equations.

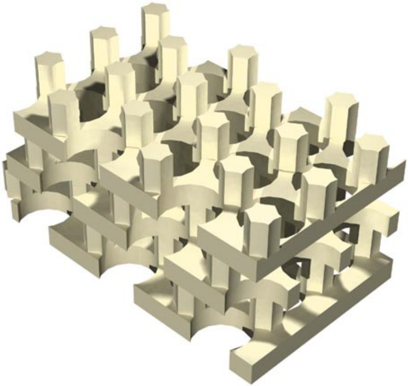
Part 1
Photonic Crystals

Part 2
Photonic Topological Insulators

Part 3
Effective Models

Part 1
Photonic Crystals

Photonic Cyrstals



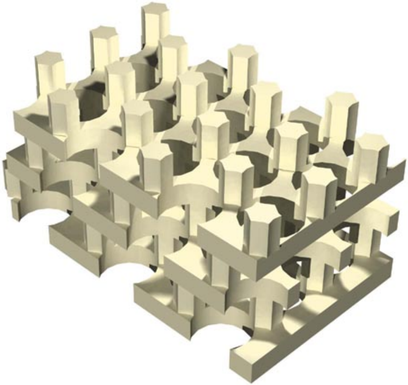
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. $W^* = W$ *(lossless)*
2. $0 < c\,\mathbf{1} \leq W \leq C\,\mathbf{1}$ *(excludes negative index mat.)*
3. *W frequency-independent (response instantaneous)*
4. *W periodic wrt lattice* $\Gamma \simeq \mathbb{Z}^3$

Photonic Cyrstals



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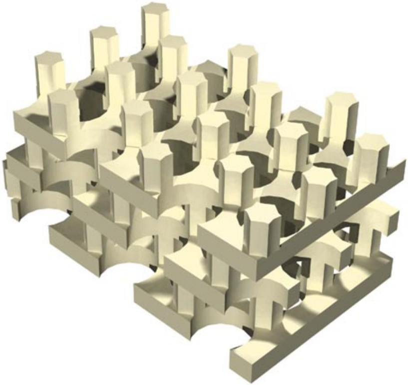

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Symmetries of Ordinary Materials

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\varepsilon & 0 \\ 0 & \mathrm{Re}\,\mu \end{pmatrix}, \qquad \varepsilon \not\propto \mu$$

3 symmetries: $U\,W = W\,U$ where $U =$

1. $C : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, \overline{\mathbf{H}})$ complex conjugation
   relies on $\varepsilon, \mu, \chi \in \mathbb{R}$, "real fields remain real"
2. $J : (\mathbf{E}, \mathbf{H}) \mapsto (\mathbf{E}, -\mathbf{H})$ implements time-reversal
   relies on $\chi = 0$
3. $T = JC : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, -\overline{\mathbf{H}})$ implements time-reversal

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Symmetries of Ordinary Materials

These **3 symmetries can be broken separately**!

Part 2

Photonic Topological Insulators

Quantum-Light Analogies

»A photonic crystal is to light what a crystalline solid is to an electron.«

Photonic Topological Insulators

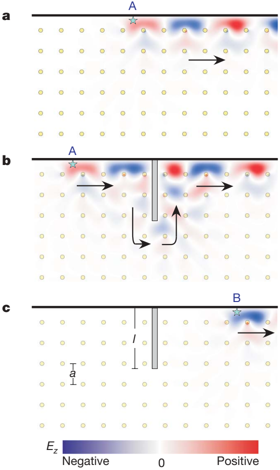
1987-2005 Research focuses on photonic crystals with *photonic band gap*

2005-now Two seminal work by *Onoda, Murakami & Nagaosa* as well as *Raghu & Haldane*: study of *topological* properties

Topologically Protected Edge Modes

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}\right\} \Longrightarrow$$

symmetry breaking



Joannopoulos, Soljačić et al (2009)

Classification of Topological Insulators in QM

Cartan-Altland-Zirnbauer classification scheme

Classifies according to discrete symmetries

- 3 types of (pseudo) symmetries:
  U unitary/antiunitary, $U^2 = \pm \mathrm{id}$,

$U\,H(k)\,U^{-1} = +H(-k)$ time-reversal symmetry ($\pm$TR)

$U\,H(k)\,U^{-1} = -H(-k)$ particle-hole (pseudo) symmetry ($\pm$PH)

$U\,H(k)\,U^{-1} = -H(+k)$ chiral (pseudo) symmetry (χ)

- 10 CAZ classes
- Relies on $\mathrm{i}\partial_t \psi = H\psi$ (Schrödinger equation)

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Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}\big(\mathbf{E},\mathbf{H}\big) = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

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1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_w(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

$$\big\| (\mathbf{E}, \mathbf{H}) \big\|^2_{L^2_w} = \int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

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$$\big\| (\mathbf{E}, \mathbf{H}) \big\|^2_{L^2_w} = 2\, \mathcal{E}(\mathbf{E}, \mathbf{H})$$

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$$J_W = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in L^2_W(\mathbb{R}^3, \mathbb{C}^6) \;\middle|\; \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0 \right\}$$

Schrödinger Formalism of the Maxwell Equations

1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_w(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

$$\big\| (\mathbf{E}, \mathbf{H}) \big\|^2_{L^2_w} = \big\langle (\mathbf{E}, \mathbf{H}), W(\mathbf{E}, \mathbf{H}) \big\rangle_{L^2(\mathbb{R}^3, \mathbb{C}^6)}$$

2. *Dynamical equations* $\rightsquigarrow$ »Schrödinger equation«

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Schrödinger Formalism of the Maxwell Equations

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$$\mathrm{i}\frac{\partial}{\partial t}\Psi = M\Psi$$

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This is only a **mathematical procedure**,
allows to **adapt many techniques**
initially developed for quantum mechanics
to classical electromagnetism.

The Maxwell Operator

$$M = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}$$
$$= W^{-1} \, \mathrm{Rot}$$

$M = M^*$ hermitian on *weighted* Hilbert space

$$\langle \Psi, M\Phi \rangle_w = \langle \Psi, W W^{-1} \, \mathrm{Rot}\Phi \rangle = \langle \mathrm{Rot}\Psi, \Psi \rangle$$
$$= \langle W M\Psi, \Phi \rangle = \langle M\Psi, W \Phi \rangle = \langle M\Psi, \Phi \rangle_w$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}tM}$ **unitary**, yields **conservation of energy**

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The Frequency Band Picture

$$M \cong M^{\mathcal{F}} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, M(k)$$

$$= \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_y + k)^{\times} \\ -(-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix}$$

$$\mathfrak{D}\big(M(k)\big) = \underbrace{\big(H^1(\mathbb{T}^3, \mathbb{C}^6) \cap J_w(k)\big)}_{\text{physical states}} \oplus G(k) \subset L^2_w(\mathbb{T}^3, \mathbb{C}^6)$$

$M(k)|_{G(k)} = 0 \Rightarrow$ focus on $M(k)|_{J_w(k)}$

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The Frequency Band Picture

Physical bands

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
- Bloch functions $k \mapsto \varphi_n(k)$
- both locally continuous everywhere
- both locally analytic *away from band crossings*

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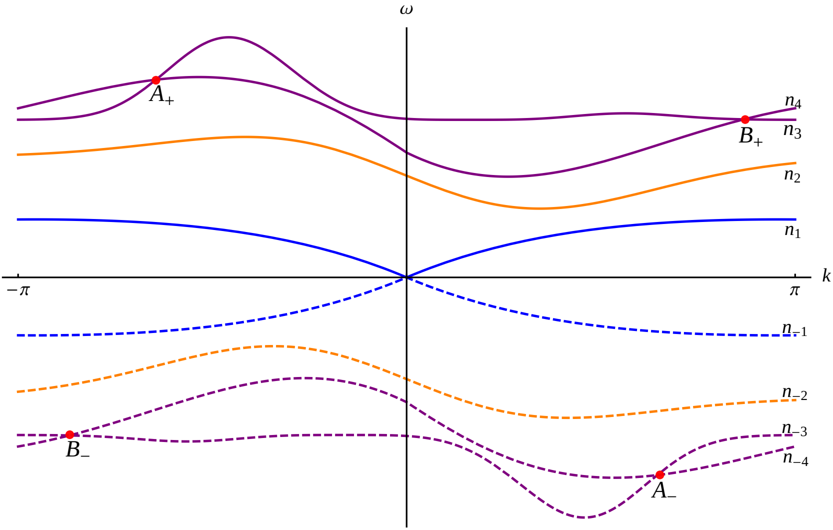

CAZ Classification of Ordinary PhCs

Symmetry	Action	Classified as	Physical meaning
C	$C\,M(k)\,C = -M(-k)$	+PH	"real states remain real"
$J = \sigma_3 \otimes \mathrm{id}$	$J\,M(k)\,J = -M(+k)$	χ	implements time-reversal
$T = J\,C$	$T\,M(k)\,T = +M(-k)$	+TR	implements time-reversal

$\Rightarrow$ Ordinary PhCs are of class BDI

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Comparison Between Photonics and Quantum Mechanics

Material	Photonics	Quantum mechanics
ordinary	class BDI +PH, +TR, χ	class AI +TR
exhibiting edge currents	class AIII χ	class A/AII none/-TR

Important consequences

- Class BDI ***not* topologically trivial**
  (also relevant in theory of topological superconductors)
- Existing derivations of topological effects in crystalline solids **do not automatically apply** to photonic crystals

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What about **other symmetries**?

Symmetries of Maxwell Operator in Matter

Product structure of $M = W^{-1}\,\mathrm{Rot}$:

$$\left.\begin{aligned} U\,\mathrm{Rot}\,U^{-1} &= \pm\mathrm{Rot} \\ U\,W\,U^{-1} &= \pm W \end{aligned}\right\} \Longrightarrow \quad U\,M\,U^{-1} = \pm M$$

(Signs may be different)

What form do the symmetries U take?

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What form do the symmetries *U* take?

Symmetries of the Free Maxwell Operator Rot

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} = -\sigma_2 \otimes \nabla^{\times}$$

Symmetries

For $n = 1, 2, 3$

1. Complex conjugation C (antilinear)
2. $J_n = \sigma_n \otimes \mathrm{id}$ (linear)
3. $T_n = J_n \, C$ (antilinear)

Connection to symmetries in ordinary materials: $J = J_3$, $T = T_3$

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Action of symmetries on Rot

1. $C \, \mathrm{Rot} \, C = -\mathrm{Rot}$
2. $J_n \, \mathrm{Rot} \, J_n^{-1} = -\mathrm{Rot}, n = 1, 3$
   $J_2 \, \mathrm{Rot} \, J_2^{-1} = +\mathrm{Rot}$
3. $T_n \, \mathrm{Rot} \, T_n^{-1} = +\mathrm{Rot}, n = 1, 3$
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Symmetries of Maxwell Operator in Matter

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(Signs may be different)

Symmetries $U = T_n, C, J_n$, $n = 1, 2, 3$

CAZ \ realized			
A	*none*		
AIII	$J_1 \equiv \chi$	$J_2 \equiv \chi$	$J_3 \equiv \chi$
AI	$T_1 \equiv +\mathrm{TR}$	$T_3 \equiv +\mathrm{TR}$	$C \equiv +\mathrm{TR}$
AII	$T_2 \equiv -\mathrm{TR}$		
D	$T_1 \equiv +\mathrm{PH}$	$T_3 \equiv +\mathrm{PH}$	$C \equiv +\mathrm{PH}$
C	$T_2 \equiv -\mathrm{PH}$		

CAZ \ realized			
BDI	$T_1 \equiv +\mathrm{TR}$ $C \equiv +\mathrm{PH}$	$C \equiv +\mathrm{TR}$ $T_1 \equiv +\mathrm{PH}$	$T_3 \equiv +\mathrm{TR}$ $C \equiv +\mathrm{PH}$
BDI	$C \equiv +\mathrm{TR}$ $T_3 \equiv +\mathrm{PH}$	$T_3 \equiv +\mathrm{TR}$ $T_1 \equiv +\mathrm{PH}$	$T_1 \equiv +\mathrm{TR}$ $T_3 \equiv +\mathrm{PH}$
DIII	$T_2 \equiv -\mathrm{TR}$ $T_1 \equiv +\mathrm{PH}$	$T_2 \equiv -\mathrm{TR}$ $T_3 \equiv +\mathrm{PH}$	$T_2 \equiv -\mathrm{TR}$ $C \equiv +\mathrm{PH}$
CI	$T_1 \equiv +\mathrm{TR}$ $T_2 \equiv -\mathrm{PH}$	$T_3 \equiv +\mathrm{TR}$ $T_2 \equiv -\mathrm{PH}$	$C \equiv +\mathrm{TR}$ $T_2 \equiv -\mathrm{PH}$

Symmetries present	CAZ class	ε, μ	χ	Realized?
none	A	$\mathbb{C}$	$\mathbb{C}$	Yes
T_3	AI	$\mathbb{R}$	$\mathrm{i}\mathbb{R}$	Yes
J_3	AIII	$\mathbb{C}$	0	Yes
C	D	$\mathbb{R}$	$\mathbb{R}$	Unknown
C, J_3, T_3	BDI	$\mathbb{R}$	0	Yes

Symmetries present	CAZ class	ε, μ	χ	Realized?
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J_3	AIII	$\mathbb{C}$	0	Yes
C	D	$\mathbb{R}$	$\mathbb{R}$	Unknown
C, J_3, T_3	BDI	$\mathbb{R}$	0	Yes

Symmetries present	CAZ class	Reduced *K*-group in dimension			
		$d=1$	$d=2$	$d=3$	$d=4$
none	A	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^7$
$T_3 \equiv +\mathrm{TR}$	AI	0	0	0	$\mathbb{Z}$
$J_3 \equiv \chi$	AIII	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8$
$C \equiv +\mathrm{PH}$	D	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}^3$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}^6$
$J_3 \equiv \chi$ $C \equiv +\mathrm{PH}$	BDI	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$
$T_2 \equiv -\mathrm{PH}$ $T_3 \equiv +\mathrm{TR}$	CI	0	0	$\mathbb{Z}$	$\mathbb{Z}^4$

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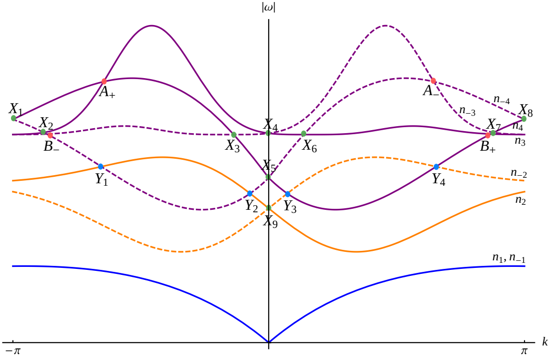
$C \equiv +\mathrm{PH}$	D	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}^3$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}^6$
$J_3 \equiv \chi$ $C \equiv +\mathrm{PH}$	BDI	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$
$T_2 \equiv -\mathrm{PH}$ $T_3 \equiv +\mathrm{TR}$	CI	0	0	$\mathbb{Z}$	$\mathbb{Z}^4$

Part 3
Effective Models

Tight-Binding Models from Ad Hoc Considerations



1. **Obtain band spectrum** by solving a *second*-order equation for electric/magnetic field only, e. g. $M(k)^2_{EE}\,\varphi^E_n(k) = \lambda_n(k)^2\,\varphi^E_n(k)$
2. **Pick a family of bands**, e. g. with a **conical intersection** (A_+, Y_1)
3. Use a **graphene-type tight-binding model** to understand light propagation for states located near intersection

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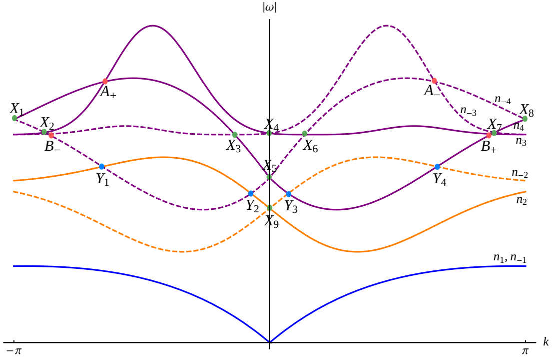

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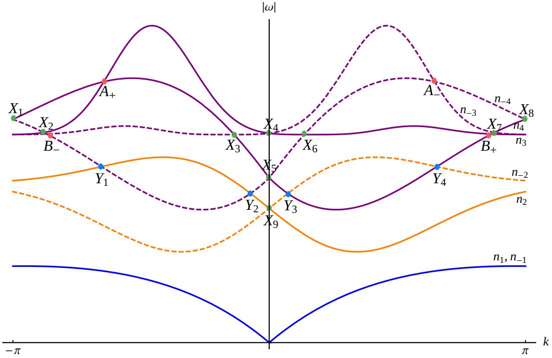
Tight-Binding Models from Ad Hoc Considerations



1. **Obtain band spectrum** by solving a ***second*-order equation** for electric/magnetic field only, e. g. $M(k)^2_{EE}\,\varphi^E_n(k) = \lambda_n(k)^2\,\varphi^E_n(k)$
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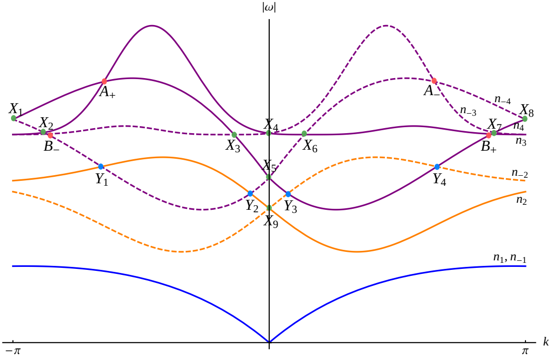

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Caution!

Procedure yields tight-binding operator M_{eff}

Problems

1. Connection of M_{eff} to dynamics?
2. Nature of symmetries?
3. Correct notion of Berry connection?

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First- vs. Second-Order Framework

Assume $\chi = 0$ (no bianisotropy).

First- vs. Second-Order Framework

first order *second order*

$$\mathrm{i}\partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \quad \Longleftrightarrow \quad \left(\partial_t^2 + M^2\right) \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

$$M = \begin{pmatrix} 0 & +\mathrm{i}\,\varepsilon^{-1}\,\nabla^{\times} \\ -\mathrm{i}\,\mu^{-1}\,\nabla^{\times} & 0 \end{pmatrix} \quad \Longrightarrow \quad M^2 = \begin{pmatrix} \varepsilon^{-1}\,\nabla^{\times}\,\mu^{-1}\,\nabla^{\times} & 0 \\ 0 & \mu^{-1}\,\nabla^{\times}\,\varepsilon^{-1}\,\nabla^{\times} \end{pmatrix}$$

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k) \quad \Longrightarrow \quad M(k)^2\,\varphi_n(k) = \big(\omega_n(k)\big)^2\,\varphi_n(k)$$

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M block-offdiagonal	$\Longrightarrow$	M^2 block-diagonal
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Compute frequency bands *starting* from

$$M(k)^2_{EE}\varphi^E_n(k) = \big(\lambda_n(k)\big)^2\,\varphi^E_n(k)$$

Assumption $\lambda_n(k) \geq 0 \Longrightarrow$ yields $|\omega|$ spectrum

$\rightsquigarrow$ Sign important for dynamics!

$$0 = \big(\partial_t^2 + M(k)^2\big)\begin{pmatrix}\mathbf{E}\\\mathbf{H}\end{pmatrix} = \big(\partial_t + \mathrm{i}\,M(k)\big)\,\big(\partial_t - \mathrm{i}\,M(k)\big)\begin{pmatrix}\mathbf{E}\\\mathbf{H}\end{pmatrix}$$

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ω spectrum vs. $|\omega|$ spectrum

First-order formulation

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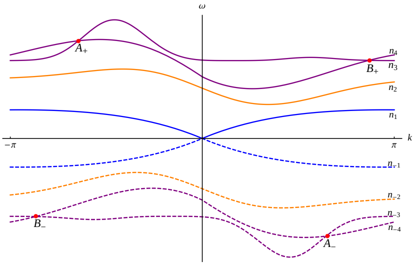
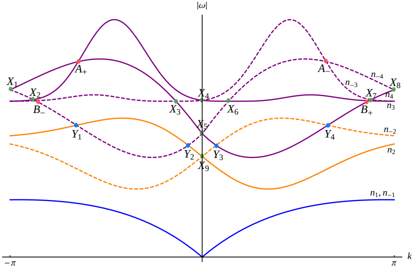
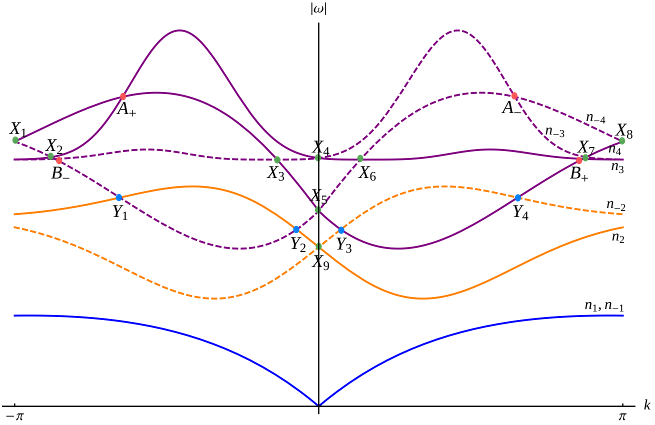


Second-order formulation

$$M(k)^2\varphi_n(k) = |\omega_n(k)|^2\ \varphi_n(k)$$



ω spectrum vs. $|\omega|$ spectrum



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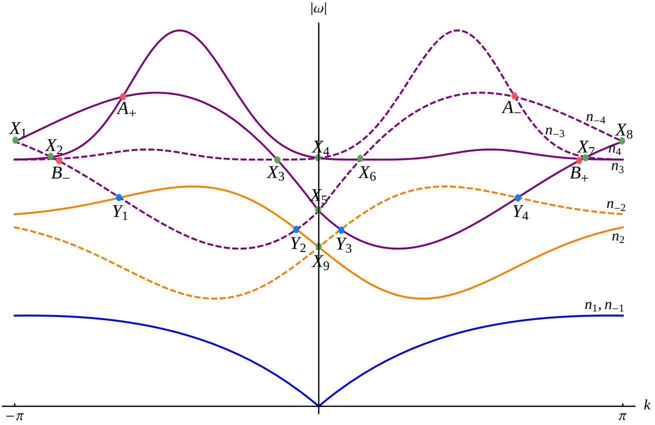

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Symmetries

Classification of (anti-)unitary U with $U^2 = \pm \mathrm{id}$ with

$$U \, M(k)^2 \, U^{-1} = M(\pm k)^2$$

in Cartan-Altland-Zirnbauer scheme, e. g.

$$\left. \begin{array}{r} C \, M(k) \, C = -M(-k) \\ \Rightarrow \; C \, M(k)^2 \, C = +M(-k)^2 \end{array} \right\} \quad \text{vs.} \quad \left\{ \begin{array}{r} T \, M(k) \, T = +M(-k) \\ \Rightarrow \; T \, M(k)^2 \, T = +M(-k)^2 \end{array} \right.$$

$\Rightarrow$ No way to distinguish PH and TR symmetry!
Ditto for chiral vs. proper symmetry

$\Rightarrow$ CAZ classification **impossible** in second-order framework!

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Proper definition of the Berry Connection

$$\begin{aligned}\mathcal{A}(k) &= \mathrm{i}\left\langle \varphi_n(k), \nabla_k \varphi_n(k)\right\rangle_w = \mathrm{i}\left\langle \varphi_n(k), W\,\nabla_k \varphi_n(k)\right\rangle \\ &= \mathrm{i}\left\langle \varphi_n^E(k), \varepsilon\,\nabla_k \varphi_n(k)\right\rangle + \mathrm{i}\left\langle \varphi_n^H(k), \mu\,\nabla_k \varphi_n^H(k)\right\rangle\end{aligned}$$

- Berry connection sometimes computed using only $\varphi_n^E(k)$
- However: $\left\|\mathbf{E}(t)\right\|_{\varepsilon}^2 = \left\langle \mathbf{E}(t), \varepsilon\,\mathbf{E}(t)\right\rangle$ *not* conserved quantity!
- $\Rightarrow \mathcal{A}^E(k) = \mathrm{i}\left\langle \varphi_n^E(k), \varepsilon\,\nabla_k \varphi_n^E(k)\right\rangle$ *not* a connection
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Effective Tight-Binding Models

Goal: Find

1. an orthogonal projection P and
2. a **simpler** effective operator M_{eff}
   (equivalent to a tight-binding operator)

so that for states from ran P we have

$$\mathrm{e}^{-\mathrm{i}tM} P = \mathrm{e}^{-\mathrm{i}tM_{\mathrm{eff}}} P + \text{error}.$$

Effective Models Should Retain All Symmetries!

For *topological* effects: ***M* and *M*$_{\mathrm{eff}}$** which enter

$$\mathrm{e}^{-\mathrm{i}tM} P = \mathrm{e}^{-\mathrm{i}tM_{\mathrm{eff}}} P + \mathrm{error}$$

should be in the same CAZ class

- *M* and *M*$_{\mathrm{eff}}$ possess the same number and type of symmetries
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- CMP paper explains how to ***compute* effective tight-binding operators** in the presence of adiabatic perturbations.
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Conclusion

Covered in the talk today

Part 2

Classification of photonic topological insulators

- Schrödinger formalism of electromagnetism
  ⇝ application of CAZ scheme for TIs
- Complete classification table in publication
- Ordinary material in class BDI (3 symmetries)
  ⇝ different from time-reversal-invariant quantum systems!
  ⇝ each symmetry can be broken individually

Part 3

Effective light dynamics

- For topological effects: M and M_{eff} of same CAZ class
- For adiabatic perturbations: explicit form of corrections available
- Ray optics equations also available

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1. Realize many effects for light at **optical** frequencies.
   $\rightsquigarrow$ Necessary for integration with optical devices
2. Rely as much as possible on **ordinary** materials.
   $\rightsquigarrow$ Ordinary materials in non-trivial topological class!
3. Include **non-linear** effects.
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Thank you for your attention!

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Part 4
Encore

Open Problems

- Better understanding of topological classes BDI and AIII
  $\rightsquigarrow$ also relevant for topological superconductors
- Effective dynamics for classes BDI, D and AIII $\rightsquigarrow$ edge currents
- Bulk-edge correspondences
  $\rightsquigarrow$ photonic analog of transverse conductivity?
- Effects of **non-linearity** $\rightsquigarrow$ topological solitons?
- Persistence of edge currents in presence of **random** impurities
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Relevance of symmetries for classification

Mathematically irrelevant symmetries, e. g.

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