Photonic Topological Insulators and Their Effective Dynamics

in collarboration with Giuseppe De Nittis

Max Lein
AIMR
2015.11.02@RIKEN

Talk Based on

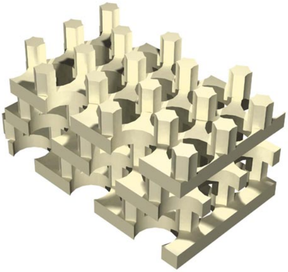
Collaboration with **Giuseppe De Nittis**

- *On the Role of Symmetries in the Theory of Photonic Crystals*
  Annals of Physics **350**, pp. 568–587, 2014
- *Effective Light Dynamics in Perturbed Photonic Crystals*
  Comm. Math. Phys. **332**, issue 1, pp. 221–260, 2014
- *Derivation of Ray Optics Equations in Photonic Crystals Via a Semiclassical Limit*
  arxiv:1502.07235, submitted for publication, 2015

Periodic Light Conductors

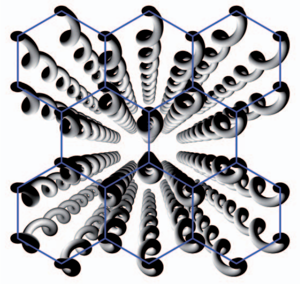
Photonic Crystals

Johnson & Joannopoulos (2004)



Periodic Waveguide Arrays

Rechtsman, Szameit et al (2013)



- periodic structure $\Longrightarrow$ *peculiar light conduction properties*
- natural photonic crystals: gem stones, beetle shells, butterfly wings, chameleon
- artificial PLCs can be *engineered arbitrarily* and *inexpensively*
- *"band structure engineering"*
  $\rightsquigarrow$ **photonic band gaps**, slow light, low-dispersion materials

A Novel Class of Materials: *Photonic Topological Insulators*

Theory

Predicted by

- Onoda, Murakami and Nagaosa (2004)
- Raghu and Haldane (2005)

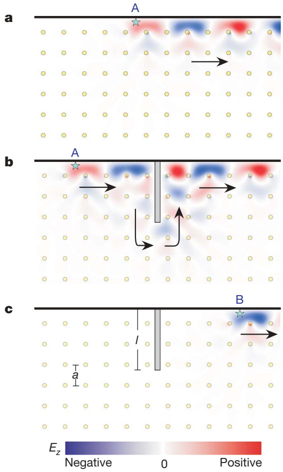
Experiment

... and realized in

- 2d **photonic crystals** for **microwaves** by Joannopoulos, Soljačić et al (2009)
- **periodic waveguide arrays** for light at **optical frequencies** by Rechtsman, Szameit et al (2013)

A Novel Class of Materials: *Photonic Topological Insulators*

$$\left.\begin{array}{c}\begin{pmatrix}\overline{\varepsilon} & 0 \\ 0 & \overline{\mu}\end{pmatrix} \neq \begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix} \\ \text{symmetry breaking}\end{array}\right\} \Longrightarrow$$



Joannopoulos, Soljačić et al (2009)

Trends in Research on Photonics

1. Realize many effects for light at **optical** frequencies.
   $\rightsquigarrow$ Necessary for integration with electronic devices
2. Include **topological** effects.
3. Rely as much as possible on **ordinary** materials.
   $\rightsquigarrow$ Ordinary materials in non-trivial topological class!
4. Include **non-linear** effects.
   $\rightsquigarrow$ Should be particularly **strong in topological edge modes** (remain localized!)

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Part 1
Schrödinger Formalism of Light

Part 2
A Primer on Topological Insulators

Part 3
Photonic Topological Insulators

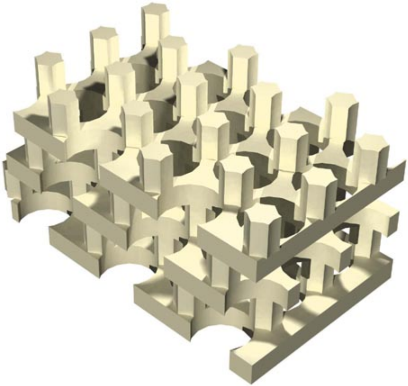
Part 4
Effective Models

Part 1

Schrödinger Formalism of Light

This is only a **mathematical procedure**, allows to **adapt many techniques** initially developed for quantum mechanics **to classical electromagnetism**.

Photonic Cyrstals



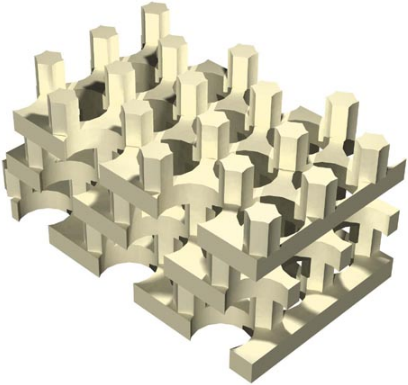
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. $W^* = W$ *(lossless)*
2. $0 < c\,\mathbf{1} \leq W \leq C\,\mathbf{1}$ *(excludes negative index mat.)*
3. *W frequency-independent (response instantaneous)*
4. *W periodic wrt lattice* $\Gamma \simeq \mathbb{Z}^3$

Photonic Cyrstals



Johnson & Joannopoulos (2004)

Maxwell equations

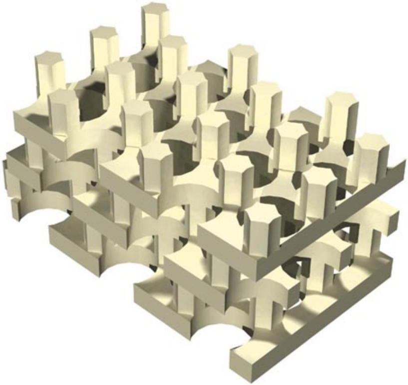
Dynamical equations

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla_x \times \mathbf{H} \\ -\nabla_x \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \nabla \cdot (\varepsilon \mathbf{E} + \chi \mathbf{H}) \\ \nabla \cdot (\chi^* \mathbf{E} + \mu \mathbf{H}) \end{pmatrix} = 0$$

Photonic Cyrstals



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Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}(\mathbf{E},\mathbf{H}) = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix}\mathbf{E}(x)\\ \mathbf{H}(x)\end{pmatrix} \cdot \begin{pmatrix}\varepsilon(x) & \chi(x)\\ \chi(x)^* & \mu(x)\end{pmatrix}\begin{pmatrix}\mathbf{E}(x)\\ \mathbf{H}(x)\end{pmatrix}$$

2. *Dynamical equations*

$$\begin{pmatrix}\varepsilon & \chi\\ \chi^* & \mu\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}-\nabla_x \times \mathbf{H}\\ +\nabla_x \times \mathbf{E}\end{pmatrix}$$

3. *No sources*

$$\begin{pmatrix}\mathrm{div} & 0\\ 0 & \mathrm{div}\end{pmatrix}\begin{pmatrix}\varepsilon & \chi\\ \chi^* & \mu\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = 0$$

Schrödinger Formalism of the Maxwell Equations

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$$\mathcal{E}\big(\mathbf{E}, \mathbf{H}\big) = \mathcal{E}\big(\mathbf{E}(t), \mathbf{H}(t)\big)$$

2. *Dynamical equations*

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Schrödinger Formalism of the Maxwell Equations

1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_w(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

$$\big\|(\mathbf{E}, \mathbf{H})\big\|^2_{L^2_w} = \int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

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3. *No sources*

$$J_w =$$

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$$\big\|(\mathbf{E}, \mathbf{H})\big\|^2_{L^2_w} = 2\,\mathcal{E}\big(\mathbf{E}, \mathbf{H}\big)$$

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$$\mathrm{i} \frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}}_{= \Psi} = \underbrace{\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}}_{= M} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

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$$\mathrm{i} \frac{\partial}{\partial t} \Psi = M \Psi$$

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3. *No sources*

$$J_w = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in L^2_w(\mathbb{R}^3, \mathbb{C}^6) \;\middle|\; \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0 \right\}$$

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2. *Dynamical equations* $\leadsto$ »Schrödinger equation«

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3. *No sources*

$$J_w = G^{\perp_w}, \qquad G = \text{gradient fields}$$

The Maxwell Operator

$$M = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^\times \\ -\mathrm{i}\nabla^\times & 0 \end{pmatrix}$$
$$= W^{-1}\,\mathrm{Rot}$$

$M = M^*$ hermitian on *weighted* Hilbert space

$$\langle \Psi, M\Phi \rangle_w = \langle \Psi, W W^{-1}\,\mathrm{Rot}\Phi \rangle = \langle \mathrm{Rot}\Psi, \Psi \rangle$$
$$= \langle W M\Psi, \Phi \rangle = \langle M\Psi, W\Phi \rangle = \langle M\Psi, \Phi \rangle_w$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}tM}$ **unitary**, yields **conservation of energy**

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&= W^{-1}\,\mathrm{Rot}
\end{aligned}
$$

$M = M^*$ hermitian on *weighted* Hilbert space

$$
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\big\langle \Psi, M\Phi \big\rangle_w &= \big\langle \Psi, W W^{-1}\,\mathrm{Rot}\Phi \big\rangle = \big\langle \mathrm{Rot}\Psi, \Psi \big\rangle \\
&= \big\langle W M\Psi, \Phi \big\rangle = \big\langle M\Psi, W\Phi \big\rangle = \big\langle M\Psi, \Phi \big\rangle_w
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Quantum-Light Analogies

	Photonics	**Quantum mechanics**
$\Psi =$	em field	wave function
Hilbert space	$L^2_w(\mathbb{R}^3, \mathbb{C}^6)$	$L^2(\mathbb{R}^d)$
$\lVert\Psi\rVert^2 =$	field energy	probability
generator dynamics	Maxwell operator $M = M^* = W\,\mathrm{Rot}$	hamiltonian $H = H^* = -\Delta + V$

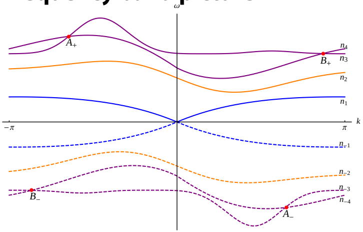
Quantum-Light Analogies

»A photonic crystal is to light what a crystalline solid is to an electron.«

Quantum-Light Analogies

photonic crystals $\longleftrightarrow$ crystalline solids

Frequency band picture



$\rightsquigarrow$ "photonic semiconductor"

Ray optics equations

Onoda et al (2004)
Raghu & Haldane (2006)
De Nittis & L. (2015)

$$\dot{r} = +\nabla_k \Omega + \lambda\, \Xi_{\mathrm{Berry}} \times \dot{k}$$
$$\dot{k} = -\nabla_r \Omega$$

$$\Omega(r, k) = \text{modified dispersion}$$

De Nittis & L. (2015): via *"semiclassical" Egorov theorem*

The Frequency Band Picture

$$M \cong M^{\mathcal{F}} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, M(k)$$
$$= \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_y + k)^{\times} \\ -(-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix}$$

$$\mathfrak{D}\big(M(k)\big) = \underbrace{\big(H^1(\mathbb{T}^3, \mathbb{C}^6) \cap J_w(k)\big)}_{\text{physical states}} \oplus G(k) \subset L^2_w(\mathbb{T}^3, \mathbb{C}^6)$$

$M(k)|_{G(k)} = 0 \Rightarrow$ focus on $M(k)|_{J_w(k)}$

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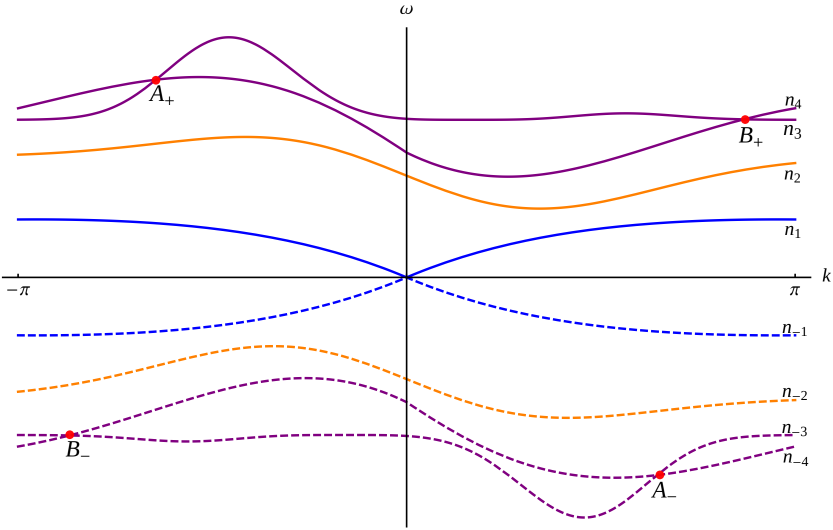
The Frequency Band Picture

Physical bands

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
- Bloch functions $k \mapsto \varphi_n(k)$
- both **locally continuous** everywhere
- both **locally analytic** *away from band crossings*

The Frequency Band Picture



Part 2

A Primer on Topological Insulators

Fundamental Notions

Altland–Zirnbauer Classification of Topological Insulators

The 10-fold way

1. **Topological class** of H $\longleftrightarrow$ Symmetries of H
2. **Phases** inside each topological class $\longleftrightarrow$ Labeled by topological invariants
3. **Bulk-edge correspondences**

Topological Classes

Symmetries of H $\longleftrightarrow$ **Topological Class** of H

- **Relies on** $\mathrm{i}\partial_t \psi = H\psi$ (Schrödinger equation)
- 3 types of (pseudo) symmetries:
  U unitary/antiunitary, $U^2 = \pm \mathrm{id}$,

  $U\,H(k)\,U^{-1} = +H(-k)$ time-reversal symmetry ($\pm$TR)

  $U\,H(k)\,U^{-1} = -H(-k)$ particle-hole (pseudo) symmetry ($\pm$PH)

  $U\,H(k)\,U^{-1} = -H(+k)$ chiral (pseudo) symmetry (χ)
- $1 + 5 + 4 = 10$ topological classes
- Physics *crucially* depends on topological class.

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Phases Inside Topological Classes

- **Inequivalent** phases inside each topological class
- *Continuous*, *symmetry-preserving* deformations of H cannot change topological phase, unless either
  – the energy gap closes (periodic case) or
  – a localization-delocalization transition happens (random case)
- Phases labeled by finite set of **topological invariants** (e. g. Chern numbers but also others)
- **Number and type** of topological invariants determined by
  – *symmetries* $\Longleftrightarrow$ topological class and
  – *dimension* of the system
- Notion that Topological Insulator $\Longleftrightarrow$ Chern number $\neq 0$ ***false!***

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Bulk-Edge Correspondences

- Properties on the boundary can be inferred from the bulk
- Consists of 3 equalities:

$$O_{\text{bulk}}(t) \approx T_{\text{bulk}}$$
$$O_{\text{edge}}(t) \approx T_{\text{edge}}$$
$$T_{\text{bulk}} = T_{\text{edge}}$$

- Number and form depends on the topological class
- Find **topological observables**

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Part 3

Photonic Topological Insulators

Symmetries of Ordinary Materials

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\varepsilon & 0 \\ 0 & \mathrm{Re}\,\mu \end{pmatrix}, \qquad \varepsilon \not\propto \mu$$

3 symmetries: $UW = WU$ where $U =$

1. $C : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, \overline{\mathbf{H}})$ complex conjugation
   relies on $\varepsilon, \mu, \chi \in \mathbb{R}$, "real fields remain real"
2. $J : (\mathbf{E}, \mathbf{H}) \mapsto (\mathbf{E}, -\mathbf{H})$ implements time-reversal
   relies on $\chi = 0$
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Symmetries of Ordinary Materials

These **3 symmetries can be broken separately**!

CAZ Classification of Ordinary PhCs

Symmetry	**Action**	**Classified as**	**Physical meaning**
C	$C\,M(k)\,C = -M(-k)$	+PH	"real states remain real"
$J = \sigma_3 \otimes \mathrm{id}$	$J\,M(k)\,J = -M(+k)$	χ	implements time-reversal
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$\Rightarrow$ Ordinary PhCs are of class BDI

Comparison Between Photonics and Quantum Mechanics

Material	Photonics	Quantum mechanics
ordinary	class BDI +PH, +TR, χ	class AI +TR
exhibiting edge currents	class AIII χ	class A/AII none/-TR

Important consequences

- Class BDI ***not* topologically trivial**
  (also relevant in theory of topological superconductors)
- Existing derivations of topological effects in crystalline solids **do not automatically apply** to photonic crystals

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What about **other symmetries**?

Symmetries of Maxwell Operator in Matter

Product structure of $M = W^{-1}\,\mathrm{Rot}$:

$$\left.\begin{aligned} U\,\mathrm{Rot}\,U^{-1} &= \pm\mathrm{Rot} \\ U\,W\,U^{-1} &= \pm W \end{aligned}\right\} \quad \Longrightarrow \quad U\,M\,U^{-1} = \pm M$$

(Signs may be different)

What form do the symmetries U take?

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Symmetries of the Free Maxwell Operator Rot

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} = -\sigma_2 \otimes \nabla^{\times}$$

Symmetries

For $n = 1, 2, 3$

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Connection to symmetries in ordinary materials: $J = J_3$, $T = T_3$

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Action of symmetries on Rot

1. $C \, \mathrm{Rot} \, C = -\mathrm{Rot}$
2. $J_n \, \mathrm{Rot} \, J_n^{-1} = -\mathrm{Rot}, n = 1, 3$
   $J_2 \, \mathrm{Rot} \, J_2^{-1} = +\mathrm{Rot}$
3. $T_n \, \mathrm{Rot} \, T_n^{-1} = +\mathrm{Rot}, n = 1, 3$
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Symmetries $U = T_n, C, J_n, n = 1, 2, 3$

CAZ \ realized			
A	*none*		
AIII	$J_1 \equiv \chi$	$J_2 \equiv \chi$	$J_3 \equiv \chi$
AI	$T_1 \equiv +\mathrm{TR}$	$T_3 \equiv +\mathrm{TR}$	$C \equiv +\mathrm{TR}$
AII	$T_2 \equiv -\mathrm{TR}$		
D	$T_1 \equiv +\mathrm{PH}$	$T_3 \equiv +\mathrm{PH}$	$C \equiv +\mathrm{PH}$
C	$T_2 \equiv -\mathrm{PH}$		

CAZ \ realized			
BDI	$T_1 \equiv +\mathrm{TR}$ $C \equiv +\mathrm{PH}$	$C \equiv +\mathrm{TR}$ $T_1 \equiv +\mathrm{PH}$	$T_3 \equiv +\mathrm{TR}$ $C \equiv +\mathrm{PH}$
BDI	$C \equiv +\mathrm{TR}$ $T_3 \equiv +\mathrm{PH}$	$T_3 \equiv +\mathrm{TR}$ $T_1 \equiv +\mathrm{PH}$	$T_1 \equiv +\mathrm{TR}$ $T_3 \equiv +\mathrm{PH}$
DIII	$T_2 \equiv -\mathrm{TR}$ $T_1 \equiv +\mathrm{PH}$	$T_2 \equiv -\mathrm{TR}$ $T_3 \equiv +\mathrm{PH}$	$T_2 \equiv -\mathrm{TR}$ $C \equiv +\mathrm{PH}$
CI	$T_1 \equiv +\mathrm{TR}$ $T_2 \equiv -\mathrm{PH}$	$T_3 \equiv +\mathrm{TR}$ $T_2 \equiv -\mathrm{PH}$	$C \equiv +\mathrm{TR}$ $T_2 \equiv -\mathrm{PH}$

Symmetries present	**CAZ class**	ε, μ	χ	**Realized?**
none	A	$\mathbb{C}$	$\mathbb{C}$	Yes
T_3	AI	$\mathbb{R}$	$\mathrm{i}\mathbb{R}$	Yes
J_3	AIII	$\mathbb{C}$	0	Yes
C	D	$\mathbb{R}$	$\mathbb{R}$	Unknown
C, J_3, T_3	BDI	$\mathbb{R}$	0	Yes
J_1, T_2, T_3	CI	$\mathbb{R}$ $\varepsilon = \mu$	$\mathrm{i}\mathbb{R}$ $\chi^* = \chi$	Yes

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J_3	AIII	$\mathbb{C}$	0	Yes
C	D	$\mathbb{R}$	$\mathbb{R}$	Unknown
C, J_3, T_3	BDI	$\mathbb{R}$	0	Yes
J_1, T_2, T_3	CI	$\mathbb{R}$ $\varepsilon = \mu$	$\mathrm{i}\mathbb{R}$ $\chi^* = \chi$	Yes

Symmetries present	**CAZ class**	**Reduced *K*-group in dimension**			
		$d = 1$	$d = 2$	$d = 3$	$d = 4$
none	A	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^7$
$T_3 \equiv +\text{TR}$	AI	0	0	0	$\mathbb{Z}$
$J_3 \equiv \chi$	AIII	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8$
$C \equiv +\text{PH}$	D	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}^3$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}^6$
$J_3 \equiv \chi$ $C \equiv +\text{PH}$	BDI	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$
$T_2 \equiv -\text{PH}$ $T_3 \equiv +\text{TR}$	CI	0	0	$\mathbb{Z}$	$\mathbb{Z}^4$

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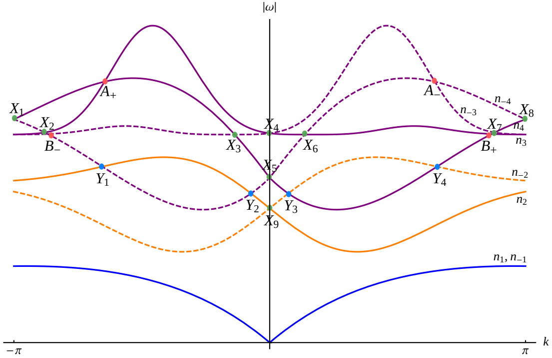
$C \equiv +$PH	D	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}^3$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}^6$
$J_3 \equiv \chi$ $C \equiv +$PH	BDI	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$
$T_2 \equiv -$PH $T_3 \equiv +$TR	CI	0	0	$\mathbb{Z}$	$\mathbb{Z}^4$

Part 4

Effective Models

Tight-Binding Models from Ad Hoc Considerations



1. **Obtain band spectrum** by solving a *second*-**order equation** for electric/magnetic field only, e. g. $M(k)^2_{EE}\,\varphi^E_n(k) = \lambda_n(k)^2\,\varphi^E_n(k)$
2. **Pick a family of bands**, e. g. with a **conical intersection** (A_+, Y_1)
3. Use a **graphene-type tight-binding model** to understand light propagation for states located near intersection

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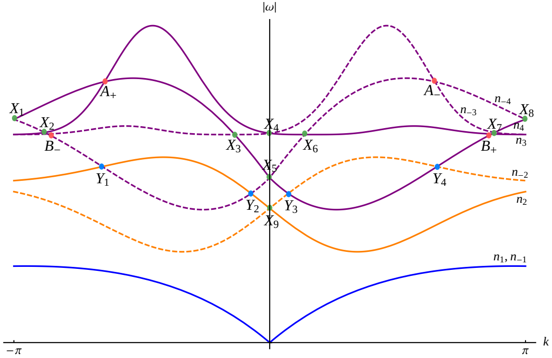

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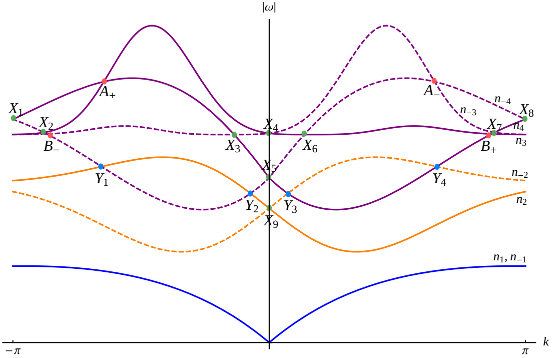

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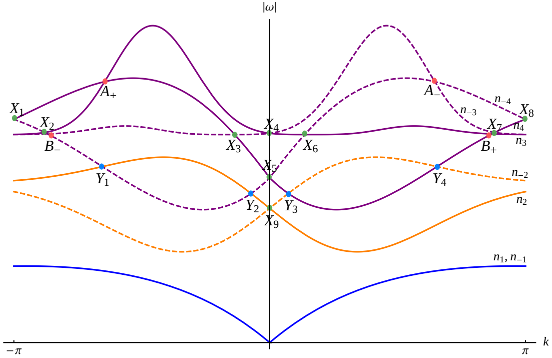

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Caution!

Procedure yields tight-binding operator M_{eff}

Problems

1. Connection of M_{eff} to dynamics?
2. Nature of symmetries?
3. Correct notion of Berry connection?

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First- vs. Second-Order Framework

Assume $\chi = 0$ (no bianisotropy).

First- vs. Second-Order Framework

first order *second order*

$$\mathrm{i}\partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \quad \Longleftrightarrow \quad \big(\partial_t^2 + M^2\big) \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

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First- vs. Second-Order Framework

Compute frequency bands *starting* from

$$M(k)^2_{EE}\varphi^E_n(k) = \big(\lambda_n(k)\big)^2 \, \varphi^E_n(k)$$

Assumption $\lambda_n(k) \geq 0 \Longrightarrow$ yields $|\omega|$ spectrum

$\rightsquigarrow$ Sign important for dynamics!

$$0 = \big(\partial_t^2 + M(k)^2\big)\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \big(\partial_t + \mathrm{i}\, M(k)\big)\,\big(\partial_t - \mathrm{i}\, M(k)\big)\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

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ω spectrum vs. $|\omega|$ spectrum

First-order formulation

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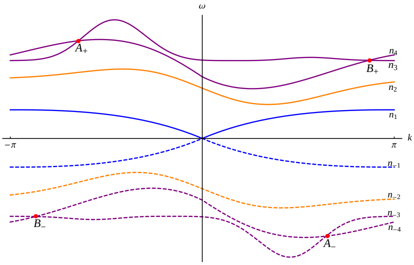
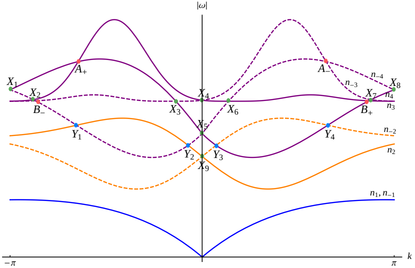


Second-order formulation

$$M(k)^2\varphi_n(k) = |\omega_n(k)|^2\,\varphi_n(k)$$



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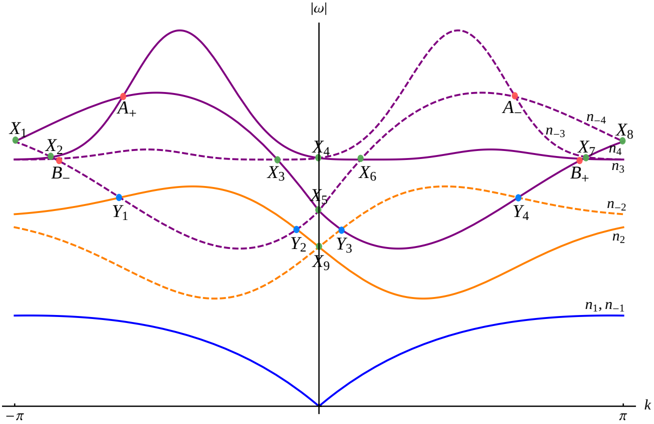

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  ⇝ eigenfunctions well-behaved at artificial crossings

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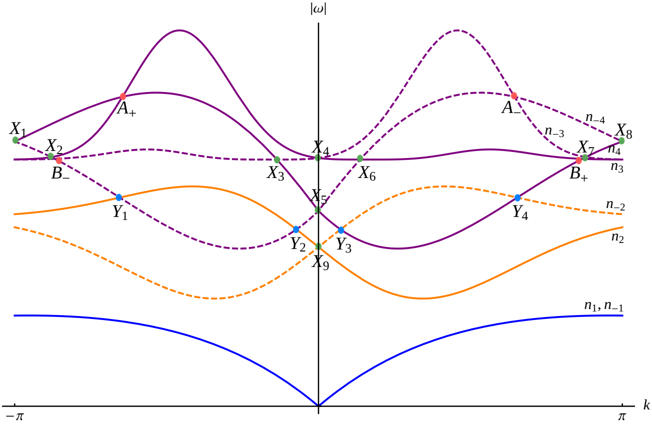

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Symmetries

Classification of (anti-)unitary U with $U^2 = \pm\mathrm{id}$ with

$$U\,M(k)^2\,U^{-1} = M(\pm k)^2$$

in Cartan-Altland-Zirnbauer scheme, e. g.

$$\left.\begin{aligned} C\,M(k)\,C &= -M(-k) \\ \Rightarrow\; C\,M(k)^2\,C &= +M(-k)^2 \end{aligned}\right\} \quad \text{vs.} \quad \left\{\begin{aligned} T\,M(k)\,T &= +M(-k) \\ \Rightarrow\; T\,M(k)^2\,T &= +M(-k)^2 \end{aligned}\right.$$

$\Rightarrow$ No way to distinguish PH and TR symmetry!
Ditto for chiral vs. proper symmetry

$\Rightarrow$ CAZ classification **impossible** in second-order framework!

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Proper definition of the Berry Connection

$$\mathcal{A}(k) = \mathrm{i} \left\langle \varphi_n(k), \nabla_k \varphi_n(k) \right\rangle_w = \mathrm{i} \left\langle \varphi_n(k), W \nabla_k \varphi_n(k) \right\rangle$$
$$= \mathrm{i} \left\langle \varphi_n^E(k), \varepsilon \nabla_k \varphi_n(k) \right\rangle + \mathrm{i} \left\langle \varphi_n^H(k), \mu \nabla_k \varphi_n^H(k) \right\rangle$$

- Berry connection sometimes computed using only $\varphi_n^E(k)$
- However: $\left\| \mathbf{E}(t) \right\|_\varepsilon^2 = \left\langle \mathbf{E}(t), \varepsilon\, \mathbf{E}(t) \right\rangle$ *not* conserved quantity!
- $\Rightarrow \mathcal{A}^E(k) = \mathrm{i} \left\langle \varphi_n^E(k), \varepsilon \nabla_k \varphi_n^E(k) \right\rangle$ *not* a connection
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Effective Tight-Binding Models

Goal: Find

1. an orthogonal projection P and
2. a **simpler** effective operator M_{eff}
   (equivalent to a tight-binding operator)

so that for states from ran P we have

$$\mathrm{e}^{-\mathrm{i}tM} P = \mathrm{e}^{-\mathrm{i}tM_{\mathrm{eff}}} P + \text{error}.$$

Effective Models Should Retain All Symmetries!

For *topological* effects: M **and** M_{eff} which enter

$$e^{-itM}\,P = e^{-itM_{\mathrm{eff}}}\,P + \text{error}$$

should be in the same CAZ class

- M and M_{eff} possess the same number and type of symmetries
- Due to misclassification of PhCs in earlier works: **disregarded** in the literature

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Conclusion

Covered in Today's Talk

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Schrödinger Formalism of Electromagnetism

- dynamical Maxwell equations $\Longleftrightarrow$ $\mathrm{i}\partial_t \Psi = M \Psi$ with $M^* = M$
  $\leadsto$ adaptation of quantum mechanical techniques to electromagnetism

Part 2
Primer on Topological Insulators

- Rests on $\mathrm{i}\partial_t \Psi = H \Psi$
- Topological classes of H $\longleftrightarrow$ symmetries of H
- 3 types of symmetries ($\pm$TR, $\pm$PH, χ)
- Phases inside of topological classes
- Bulk-edge correspondences

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- Schrödinger formalism of electromagnetism
  $\rightsquigarrow$ application of classification scheme for TIs
- Complete classification table in publication
- Ordinary material in class BDI (3 symmetries)
  $\rightsquigarrow$ different from time-reversal-invariant quantum systems!
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Effective light dynamics

- For topological effects: M and M_{eff} of same topological class
- For adiabatic perturbations: explicit form of corrections available
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  $\rightsquigarrow$ also relevant for topological superconductors
- Effective dynamics for classes BDI, D and AIII $\rightsquigarrow$ edge currents
- Bulk-edge correspondences
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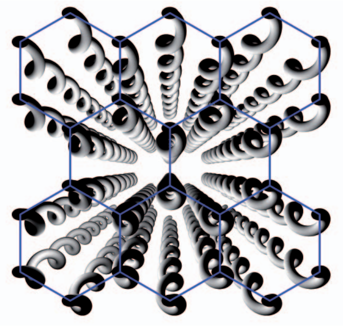
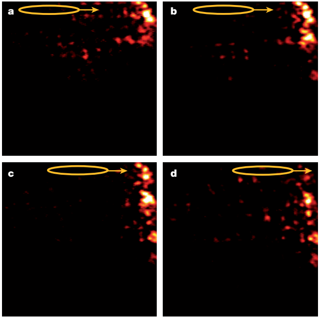
Periodic Waveguide Arrays





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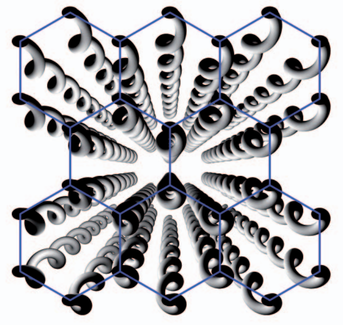

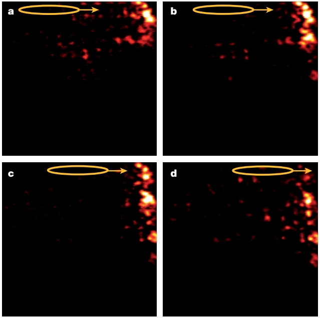

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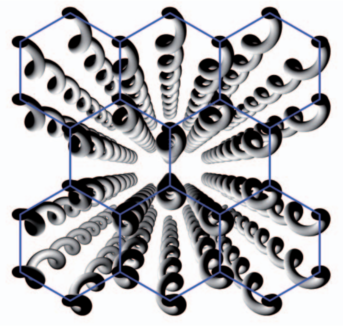

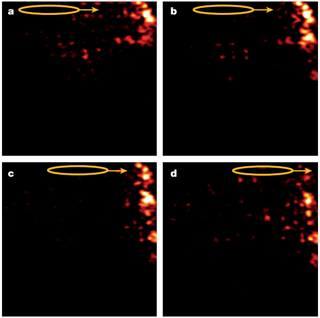

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Thank you for your attention!

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Part 5
Encore

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Mathematically irrelevant symmetries, e. g.

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