Classification of Photonic Topological Insulators and Their Effective Dynamics

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in collarboration with Giuseppe De Nittis

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2015.04.21@UOttawa

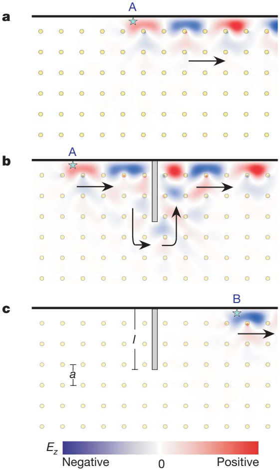
Talk Based on

Collaboration with **Giuseppe De Nittis**

- *On the Role of Symmetries in the Theory of Photonic Crystals*
  Annals of Physics **350**, pp. 568--587, 2014
- *Effective Light Dynamics in Perturbed Photonic Crystals*
  Comm. Math. Phys. **332**, issue 1, pp. 221--260, 2014
- *Derivation of Ray Optics Equations in Photonic Crystals Via a Semiclassical Limit*
  arxiv:1502.07235, submitted for publication, 2015

Long-Term Goal

$$\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \quad \Longrightarrow$$



Wang et al (2009)

Long-Term Goal

Understand ***how topological effects emerge from electrodynamics****,*
starting from Maxwells equations.

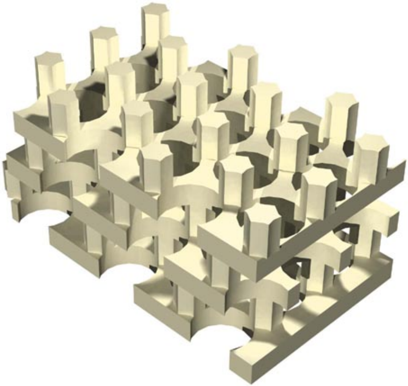
Part 1
Photonic Crystals

Part 2
Photonic Topological Insulators

Part 3
Effective Models

Part 1
Photonic Crystals

Photonic Cyrstals



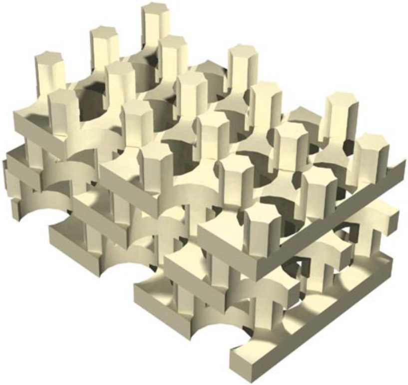
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. $0 < c\,\mathbf{1} \leq W \leq C\,\mathbf{1}$ *(excludes metamaterials)*
2. $W^* = W$ *(lossless)*
3. *W frequency-independent (response instantaneous)*
4. *W periodic wrt lattice* $\Gamma \simeq \mathbb{Z}^3$

Photonic Cyrstals



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla_x \times \mathbf{H} \\ +\nabla_x \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Symmetries of Ordinary Materials

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\varepsilon & 0 \\ 0 & \mathrm{Re}\,\mu \end{pmatrix}, \qquad \varepsilon \not\propto \mu$$

3 symmetries

1. $C : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, \overline{\mathbf{H}})$ complex conjugation
   relies on $\varepsilon, \mu, \chi \in \mathbb{R}$, "real fields remain real"
2. $T : (\mathbf{E}, \mathbf{H}) \mapsto (\mathbf{E}, -\mathbf{H})$ implements time-reversal
   relies on $\chi = 0$
3. $J = TC : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, -\overline{\mathbf{H}})$ implements time-reversal

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Symmetries of Ordinary Materials

Each of these **3 symmetries can be broken separately**!

Part 2

Photonic Topological Insulators

Quantum-Light Analogies

»A photonic crystal is to light what a crystalline solid is to an electron.«

Photonic Topological Insulators

1987-2005 Research focuses on photonic crystals with *photonic band gap*

2005-now Two seminal work by *Onoda, Murakami & Nagaosa* as well as *Raghu & Haldane*: study of *topological* properties

Topologically Protected Edge Modes

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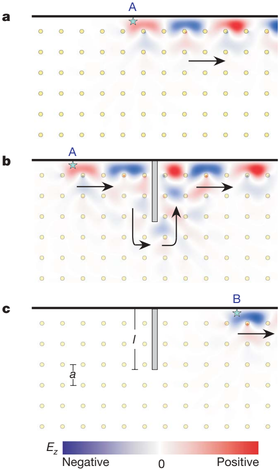

Wang et al (2009)

Classification of Topological Insulators in QM

Cartan-Altland-Zirnbauer classification scheme

Classifies according to discrete symmetries

- 3 types of (pseudo) symmetries:
  U unitary/antiunitary, $U^2 = \pm\mathrm{id}$,

$U\,H(k)\,U^{-1} = +H(-k)$	time-reversal symmetry ($\pm$TR)
$U\,H(k)\,U^{-1} = -H(-k)$	particle-hole (pseudo) symmetry ($\pm$PH)
$U\,H(k)\,U^{-1} = -H(+k)$	chiral (pseudo) symmetry (χ)

- 10 CAZ classes
- Relies on $\mathrm{i}\partial_t\psi = H\psi$ (Schrödinger equation)

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Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}(\mathbf{E},\mathbf{H}) = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

2. *Dynamical equations*

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla_x \times \mathbf{H} \\ +\nabla_x \times \mathbf{E} \end{pmatrix}$$

3. *No sources*

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}\big(\mathbf{E}, \mathbf{H}\big) = \mathcal{E}\big(\mathbf{E}(t), \mathbf{H}(t)\big)$$

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1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_w(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

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1. *Field energy* $(\mathbf{E},\mathbf{H}) \in L^2_w(\mathbb{R}^3,\mathbb{C}^6)$ with energy norm

$$\big\|(\mathbf{E},\mathbf{H})\big\|^2_{L^2_w} = 2\,\mathcal{E}(\mathbf{E},\mathbf{H})$$

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The Maxwell Operator

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$$= W^{-1}\,\mathrm{Rot}$$

M selfadjoint on *weighted* $L^2_w(\mathbb{R}^3, \mathbb{C}^6)$
$\Rightarrow e^{-\mathrm{i}tM}$ unitary, yields conservation of energy

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$\Rightarrow \mathrm{e}^{-\mathrm{i}tM}$ unitary, yields conservation of energy

The Maxwell Operator

$$M = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}$$
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The Frequency Band Picture

$$M \cong M^{\mathcal{F}} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, M(k)$$

$$= \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_y + k)^{\times} \\ -(-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix}$$

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The Frequency Band Picture

Physical bands

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
- Bloch functions $k \mapsto \varphi_n(k)$
- both locally continuous everywhere
- both locally analytic *away from band crossings*

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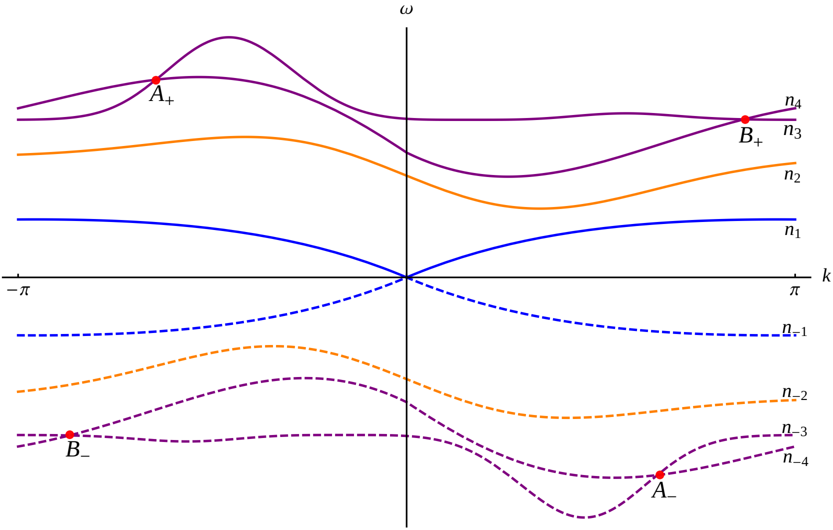

CAZ Classification of Ordinary PhCs

Symmetry	Action	Classified as	Physical meaning
C	$C\,M(k)\,C = -M(-k)$	+PH	“real states remain real”
$T = \sigma_3 \otimes \mathrm{id}$	$T\,M(k)\,T = -M(+k)$	χ	implements time-reversal
$J = T\,C$	$J\,M(k)\,J = +M(-k)$	+TR	implements time-reversal

$\Rightarrow$ Ordinary PhCs are of class BDI

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Comparison Between Photonics and Quantum Mechanics

Material	Photonics	Quantum mechanics
ordinary	class BDI +PH, +TR, χ	class AI +TR
exhibiting edge currents	class AIII χ	class A/AII none/-TR

Important consequences

- Class BDI ***not* topologically trivial**
  (also relevant in theory of topological superconductors)
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Part 3
Effective Models

Effective Tight-Binding Models

Goal: Find

1. an orthogonal projection Π and
2. a **simpler** effective operator M_{eff} (equivalent to a tight-binding operator)

so that for states from ran Π we have

$$e^{-itM}\,\Pi = e^{-itM_{\mathrm{eff}}}\,\Pi + \text{error}.$$

Effective Models Should Retain All Symmetries!

For *topological* effects: M **and** M_{eff} which enter

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should be in the same CAZ class

- M and M_{eff} possess the same number and type of symmetries
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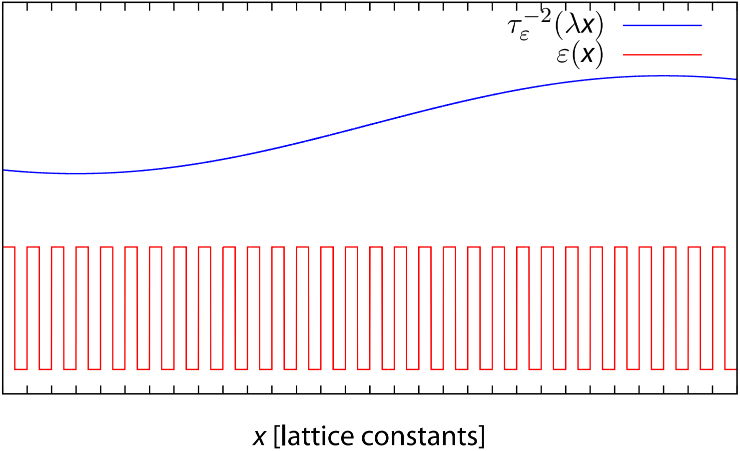
Perturbed Photonic Crystals

For simplicity

Consider PhCs of **class BDI**

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\varepsilon & 0 \\ 0 & \mathrm{Re}\,\mu \end{pmatrix}, \qquad \varepsilon \not\propto \mu$$

Perturbed Photonic Crystals



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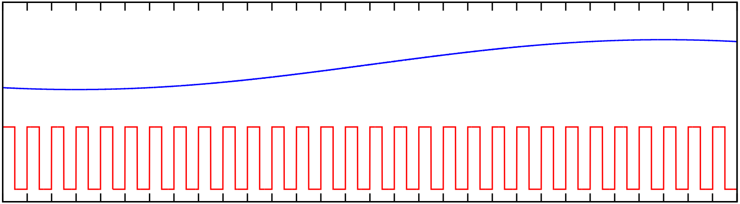

Perturbation of material constants

$$\lambda = \frac{\text{[lattice spacing]}}{\text{[length scale of modulation]}} \ll 1$$

$$\varepsilon(x) \rightsquigarrow \varepsilon_\lambda(x) := \tau_\varepsilon^{-2}(\lambda x)\, \varepsilon(x), \quad \mu(x) \rightsquigarrow \mu_\lambda(x) := \tau_\mu^{-2}(\lambda x)\, \mu(x)$$

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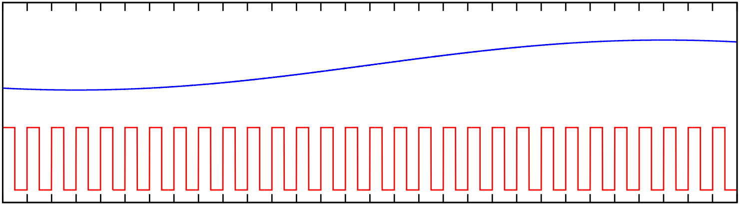

Assumption (Slow modulation)

$\tau_\varepsilon, \tau_\mu \in \mathcal{C}^\infty_{\mathsf{b}}(\mathbb{R}^3)$, $\tau_\varepsilon, \tau_\mu \geq c > 0$

Adiabatically Perturbed Maxwell Operator

$$
\begin{aligned}
M_\lambda &= S_\lambda^{-2} \, M \\
&= \begin{pmatrix} \tau_\varepsilon^2(\lambda x) & 0 \\ 0 & \tau_\mu^2(\lambda x) \end{pmatrix} \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_x)^\times \\ -(-\mathrm{i}\nabla_x)^\times & 0 \end{pmatrix}
\end{aligned}
$$

slow modulation & periodic Maxwell operator

$\rightsquigarrow$ Perturbations are multiplicative!

Existence of Physical States

Definition (Physical states: unperturbed)

$$\Pi_0 \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \; 1_{\sigma_{\mathrm{rel}}(k)}\bigl(M(k)\bigr) \text{ so that}$$

1. $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k) = \bigcup_{n \in \mathcal{I}} \{\omega_n(k)\}$ **isolated** family of bands
2. **source free**: $\mathrm{ran}\,\Pi_0 \subset J_0$
3. **real**: $C\,\Pi_0\,C = \Pi_0$

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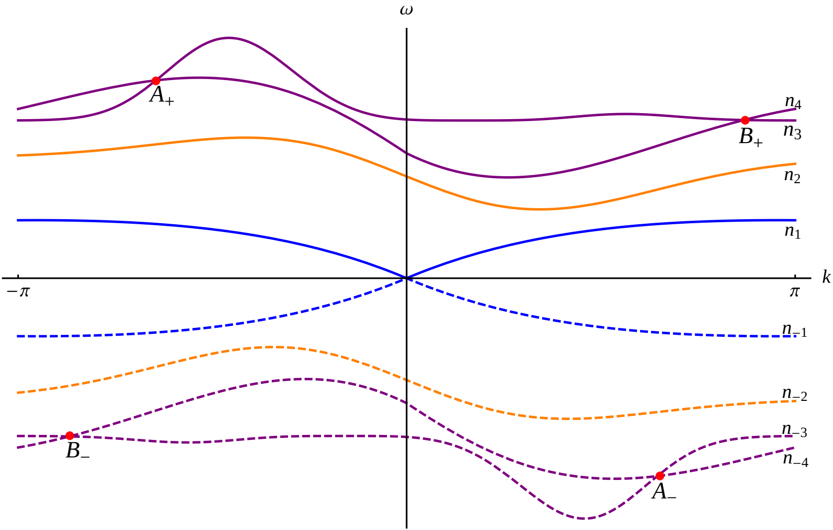

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Definition (Physical states: perturbed)

$$\Pi_\lambda \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \; 1_{\sigma_{\mathrm{rel}}(k)}\big(M(k)\big) + \mathcal{O}(\lambda) \text{ so that}$$

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Existence of Physical States

Theorem (De Nittis-L. 2014 (CMP))

Suppose the bands $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k)$ *are isolated and* $0 \notin \sigma_{\mathrm{rel}}(0)$*. Then there exist orthogonal projections*

$$\Pi_\lambda = \Pi_{+,\lambda} + \Pi_{-,\lambda} + \mathcal{O}(\lambda^\infty)$$

so that

$$\big[M_\lambda \, , \, \Pi_{\pm,\lambda}\big] = \mathcal{O}(\lambda^\infty)$$

whose ***range*** *supports* ***physical states****.*

Effective Light Dynamics

Theorem (PhCs of class BDI, De Nittis-L. 2014 (CMP))

Suppose the bands $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k)$ are isolated and $0 \notin \sigma_{\mathrm{rel}}(0)$. Then there exist a unitary V_λ and an ***effective Maxwell operator***

$$M_{\mathrm{eff}} = V_\lambda^{-1} \, \mathcal{M}_{\mathrm{eff}}\big(\mathrm{i}\lambda\nabla_k, \hat{k}\big) \, V_\lambda$$

which ***approximates the full light dynamics****,*

$$\mathrm{e}^{-\mathrm{i}tM_\lambda} \, \Pi_\lambda = \mathrm{e}^{-\mathrm{i}tM_{\mathrm{eff}}} \, \Pi_\lambda + \mathcal{O}(\lambda^\infty),$$
$$\mathrm{e}^{-\mathrm{i}tM_\lambda} \, \Pi_\lambda \, \mathrm{Re} = \mathrm{Re} \; \mathrm{e}^{-\mathrm{i}tM_{\mathrm{eff}}} \, \Pi_\lambda \, \mathrm{Re} + \mathcal{O}(\lambda^\infty),$$

and ***leaves*** ran Π_λ ***invariant*** *up to* $\mathcal{O}(\lambda^\infty)$.

Π_λ, V_λ and $\mathcal{M}_{\mathrm{eff}}$ can be computed explicitly order-by-order in λ

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Effective Light Dynamics

Corollary ("Peierl's substitution", De Nittis-L. 2014 (CMP))

For class BDI the symbol to M_{eff} associated to $\sigma_{\mathrm{rel}}(k) = \{+\omega(+k), -\omega(-k)\}$ is

$$\mathcal{M}_{\mathrm{eff}}(r,k) = \tau_{\varepsilon}(r)\,\tau_{\mu}(r)\begin{pmatrix} +\omega(+k) & 0 \\ 0 & -\omega(-k) \end{pmatrix} + \mathcal{O}(\lambda).$$

$\Rightarrow$ motivates the definition of a **Maxwell-Harper operator**

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Effective Light Dynamics

Theorem (PhCs of class D and AI, De Nittis-L. 2014 (CMP))

Similarly, we can derive effective effective dynamics of the form

$$\mathrm{e}^{-\mathrm{i}tM_\lambda}\,\Pi_\lambda = \mathrm{e}^{-\mathrm{i}tM_{\mathrm{eff}}}\,\Pi_\lambda + \mathcal{O}(\lambda^\infty)$$

if M_λ has only C-symmetry (class D) or only $J = T\,C$-symmetry (class AI).

Ingredients to the proof

Proof.

- Crucial technical tool: **pseudodifferential calculus**
- Projection onto almost-invariant subspace:

$$\Pi_\lambda = \pi_\lambda\big(\mathrm{i}\lambda\nabla_k, \hat{k}\big) + \mathcal{O}(\lambda^\infty)$$

- Unitary: $V_\lambda = U_{\mathrm{aux}} \, U_\lambda$ where

$$U_\lambda = u_\lambda\big(\mathrm{i}\lambda\nabla_k, \hat{k}\big) + \mathcal{O}(\lambda^\infty)$$

- **4 defining relations:**

$$\pi_\lambda \sharp \pi_\lambda = \pi_\lambda + \mathcal{O}(\lambda^\infty) \qquad \big[\mathcal{M}_\lambda \, , \, \pi_\lambda\big]_\sharp = \mathcal{O}(\lambda^\infty)$$

$$u_\lambda \sharp u_\lambda^* = 1 + \mathcal{O}(\lambda^\infty) \qquad u_\lambda \sharp \pi_\lambda \sharp u_\lambda^* = \pi_{\mathrm{ref}} + \mathcal{O}(\lambda^\infty)$$

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Construction of almost-invariant projection:

$$\pi_\lambda(r,k) \asymp \frac{\mathrm{i}}{2\pi} \int_{\Gamma(r,k)} \mathrm{d}z \, (\mathcal{M}_\lambda - z)^{(-1)_\sharp}(r,k)$$

- Here: $(\mathcal{M}_\lambda - z)^{(-1)_\sharp}$ is the **Moyal resolvent**

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- Existence of u_0 relies on **triviality of the Bloch bundle** ($\Leftrightarrow$ Chern numbers 0)

$$\mathcal{E}_{\mathrm{Bloch}} = \Big(\bigsqcup_{k \in \mathbb{R}^3} \operatorname{ran} \pi_0(k) \Big) / \Gamma^*$$

- u_n, $n \geq 1$, can be computed recursively and explicitly from $\{u_j\}_{0 \leq j \leq n-1}$ and $\{\pi_j\}_{0 \leq j \leq n}$

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Covered in the talk today

Part 2

Classification of photonic topological insulators

- Schrödinger formalism of electromagnetism
  $\rightsquigarrow$ application of CAZ scheme for TIs
- Complete classification table in publication
- Ordinary material in class BDI (3 symmetries)
  $\rightsquigarrow$ different from time-reversal-invariant quantum systems!
  $\rightsquigarrow$ each symmetry can be broken individually

Part 3

Effective light dynamics

- For topological effects: M and M_{eff} of same CAZ class
- So far: effective dynamics for PTIs of classes BDI, D and AI
- Explicit form of $\mathcal{O}(\lambda)$ corrections available
- Ray optics equations also available

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- Effects of **non-linearity** $\rightsquigarrow$ topological solitons?
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Thank you for your attention!

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