Existence and Absence of Non-Linear Effects in Photonic Topological Insulators

in collaboration with Giuseppe De Nittis & Maxime Gazeau

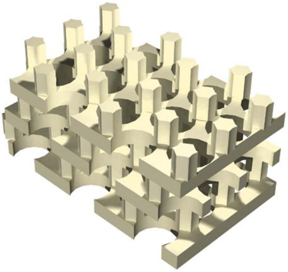
Max Lein

2015.10.06@AIMR

Periodic Light Conductors

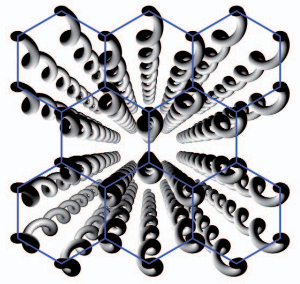
Photonic Crystals

Johnson & Joannopoulos (2004)



Periodic Waveguide Arrays

Rechtsman, Szameit et al (2013)



- periodic structure $\Longrightarrow$ *peculiar light conduction properties*
- artificial PLCs can be *engineered arbitrarily* and *inexpensively*
- *"band structure engineering"*
  $\rightsquigarrow$ **photonic band gaps**, slow light, low-dispersion materials
- natural photonic crystals: gem stones, beetle shells, butterfly wings

A Novel Class of Materials: *Photonic Topological Insulators*

Theory

Predicted by

- Onoda, Murakami and Nagaosa (2004)
- Raghu and Haldane (2005)

Experiment

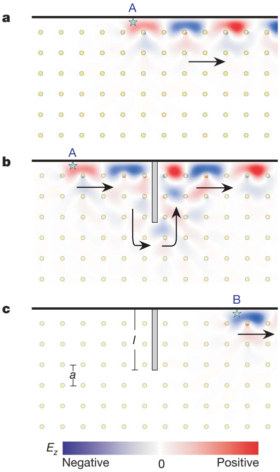
... and realized in

- 2d **photonic crystals** for **microwaves** by Joannopoulos, Soljačić et al (2009)
- **periodic waveguide arrays** for light at **optical frequencies** by Rechtsman, Szameit et al (2013)

A Novel Class of Materials: *Photonic Topological Insulators*

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \right\} \Longrightarrow$$

symmetry breaking



Joannopoulos, Soljačić et al (2009)

Trends in Research on Photonics

1. Realize many effects for light at **optical** frequencies.
   $\rightsquigarrow$ Necessary for integration with optical devices
2. Rely as much as possible on **ordinary** materials.
   $\rightsquigarrow$ Ordinary materials in non-trivial topological class!
3. Include **non-linear** effects.
   $\rightsquigarrow$ Should be particularly **strong in topological edge modes** (remain localized!)

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Long-term goal: Prove existence of **topological solitons**

Problem: Linear case not fully understood.

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Talk Based on

Collaboration with **Giuseppe De Nittis**

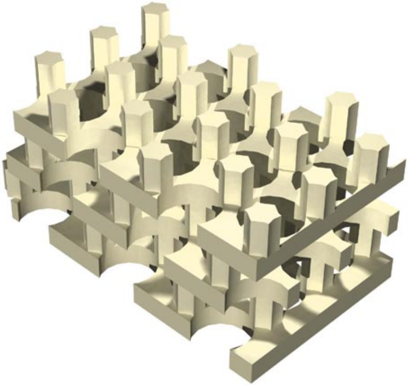
- *On the Role of Symmetries in the Theory of Photonic Crystals*
  Annals of Physics **350**, pp. 568–587, 2014

and **Maxime Gazeau**

- *Existence and Absence of Non-Linear Effects in Photonic Topological Insulators*
  in preparation, 2015

Part 1

Linear Photonic Topological Insulators

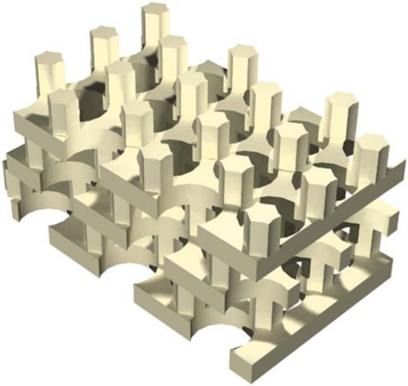


Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W_1(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. $W_1^* = W_1$ *(lossless)*
2. $0 < c\,\mathbf{1} \leq W_1 \leq C\,\mathbf{1}$ *(excludes negative index mat.)*
3. W_1 *frequency-independent* *(response instantaneous)*
4. W_1 *periodic wrt lattice* $\Gamma \simeq \mathbb{Z}^3$



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

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Express Maxwell equations in terms of $\begin{pmatrix}\mathbf{D} \\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}\varepsilon & \chi \\ \chi^* & \mu\end{pmatrix}\begin{pmatrix}\mathbf{E} \\ \mathbf{H}\end{pmatrix}$

Maxwell equations

Dynamical equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}0 & +\nabla^{\times}\\ -\nabla^{\times} & 0\end{pmatrix}\begin{pmatrix}\varepsilon & \chi\\ \chi^{*} & \mu\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}$$

Absence of sources

$$\mathrm{Div}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = 0$$

Define $\mathrm{Div} := \begin{pmatrix}\mathrm{div} & 0\\ 0 & \mathrm{div}\end{pmatrix}$

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$$\mathrm{Div}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = 0$$

Maxwell operator $M = \begin{pmatrix}0 & +\mathrm{i}\,\nabla^{\times}\\ -\mathrm{i}\,\nabla^{\times} & 0\end{pmatrix}\begin{pmatrix}\varepsilon & \chi\\ \chi^{*} & \mu\end{pmatrix}^{-1}$

Maxwell equations

Dynamical equations

$$\mathrm{i}\,\frac{\partial}{\partial t}\underbrace{\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}}_{=\Psi} = \underbrace{\begin{pmatrix}0 & +\mathrm{i}\,\nabla^{\times}\\ -\mathrm{i}\,\nabla^{\times} & 0\end{pmatrix}\begin{pmatrix}\varepsilon & \chi\\ \chi^{*} & \mu\end{pmatrix}^{-1}}_{=M}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}$$

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Dynamical equations

$$\mathrm{i}\partial_t \Psi = M\Psi$$

Absence of sources

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \in J := \ker \mathrm{Div} \subset L^2_w(\mathbb{R}^3, \mathbb{C}^6)$$

$M = M^*$ on *weighted* Hilbert space $L^2_w(\mathbb{R}^3, \mathbb{C}^6)$
Helmholtz splitting: $L^2_w(\mathbb{R}^3, \mathbb{C}^6) = \ker \mathrm{Div} \oplus W_1 \,\mathrm{ran}\,\mathrm{Grad}$

Maxwell equations

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Symmetries of Ordinary Materials

$$W_1^{-1} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\varepsilon & 0 \\ 0 & \mathrm{Re}\,\mu \end{pmatrix}, \qquad \varepsilon \not\propto \mu$$

3 symmetries: $V W_1 = W_1 V$ where $V =$

1. $C : (\mathbf{D}, \mathbf{B}) \mapsto (\overline{\mathbf{D}}, \overline{\mathbf{B}})$ complex conjugation
   relies on $\varepsilon, \mu, \chi \in \mathbb{R}$, "real fields remain real"
2. $J : (\mathbf{D}, \mathbf{B}) \mapsto (\mathbf{D}, -\mathbf{B})$ implements time-reversal
   relies on $\chi = 0$
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Symmetries of Ordinary Materials

These **3 symmetries can be broken separately**!

Symmetries & Classification of PTIs

Cartan-Altland-Zirnbauer classification scheme

Classifies according to discrete symmetries

- **3 types of (pseudo) symmetries:**
  V unitary/antiunitary, $V^2 = \pm\mathrm{id}$,

$$V\,H(k)\,V^{-1} = +H(-k) \quad \text{time-reversal symmetry } (\pm\text{TR})$$
$$V\,H(k)\,V^{-1} = -H(-k) \quad \text{particle-hole (pseudo) symmetry } (\pm\text{PH})$$
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- 10 CAZ classes
- Relies on $\mathrm{i}\partial_t\psi = H\psi$ (Schrödinger equation)

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CAZ Classification of Ordinary PhCs

Symmetry	**Action**	**Classified as**	**Physical meaning**
C	$C\,M(k)\,C = -M(-k)$	+PH	"real states remain real"
$J = \sigma_3 \otimes \mathrm{id}$	$J\,M(k)\,J = -M(+k)$	χ	implements time-reversal
$T = J\,C$	$T\,M(k)\,T = +M(-k)$	+TR	implements time-reversal

$\Rightarrow$ Ordinary PhCs are of class BDI

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$\Rightarrow$ Ordinary PhCs are of class BDI

Symmetries & Classification of PTIs

Material	**Photonics**	**Quantum mechanics**
ordinary	class BDI +PH, +TR, χ	class AI +TR
exhibiting edge currents	class AIII χ	class A/AII none/-TR

G. De Nittis & M. L., Annals of Physics **350**, pp. 568–587, 2014

Important consequences

- Class BDI ***not* topologically trivial**
  (also relevant in theory of topological superconductors)
- Existing derivations of topological effects in crystalline solids **do not automatically apply** to photonic crystals

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The Frequency Band Picture

$$M \cong M^{\mathcal{F}} = \int_{\mathbb{T}^*}^{\oplus} \mathrm{d}k \, M(k)$$
$$\cong \int_{\mathbb{T}^*}^{\oplus} \mathrm{d}k \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_y + k)^{\times} \\ -(-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1}$$

Frequency bands & Bloch functions

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
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- both locally continuous everywhere
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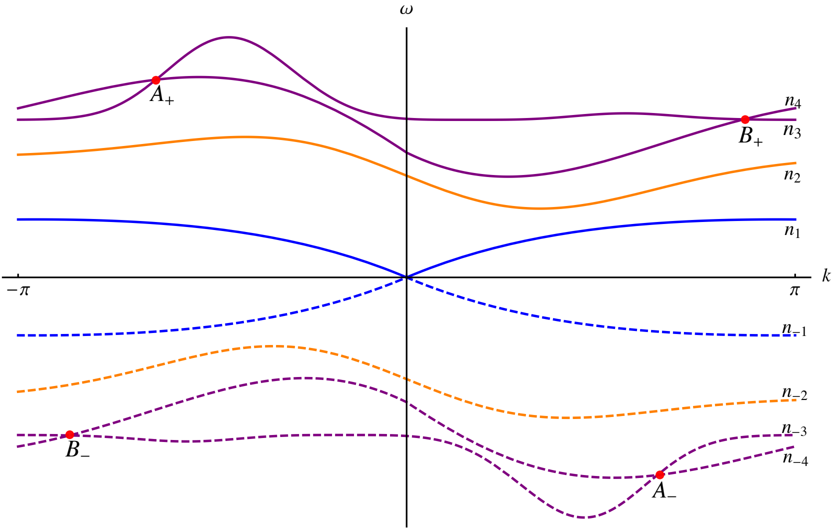

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Parity $(P\Psi)(x) := \Psi(-x)$ relevant for non-linear interactions only
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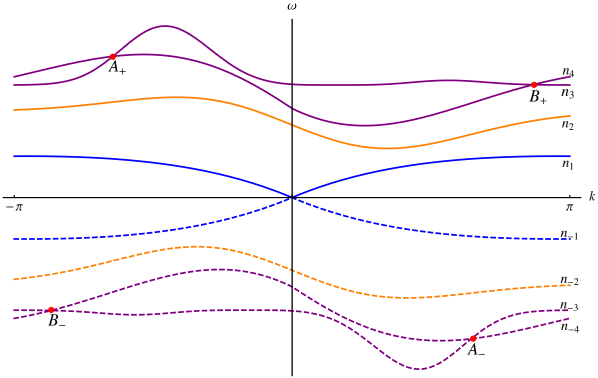
Relevant symmetries for topological classification

The Bloch Vector Bundle

Relevant family of bands $\sigma_{\mathrm{rel}}(k) := \bigcup_{n\in\mathcal{I}}\{\omega_n(k)\}$

$|\mathcal{I}| < \infty$ and $\sigma_{\mathrm{rel}}(k)$ separated by a local **spectral gap**.



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Definition (Bloch Bundle)

The Bloch bundle $\mathcal{E}_{\text{B}} = \big(\xi_{\text{B}}, \mathbb{T}^*, \pi\big)$ associated to $\sigma_{\text{rel}}(k)$ is defined as

$$\xi_{\text{B}} := \bigsqcup_{k \in \mathbb{T}^*} \text{span}\,\{\varphi_n(k)\}_{n \in \mathcal{I}} \xrightarrow{\;\pi\;} \mathbb{T}^*.$$

The Bloch Vector Bundle

Assumption (Symmetric choice of bands)

For all discrete symmetries V with $V\,M(k)\,V^{-1} = \pm M\big((-1)^s k\big)$ we have

$$V \,\mathrm{span}\,\big\{\varphi_n\big((-1)^s k\big)\big\}_{n\in\mathcal{I}} = \mathrm{span}\,\big\{\varphi_n(k)\big\}_{n\in\mathcal{I}}.$$

Lemma

The Bloch bundle $\mathcal{E}_{\mathrm{B}}$ is a class X vector bundle where X = A, AI, AII, AIII is the topological class of the linear Maxwell operator M.

Remark

- *Presently only X-vector bundles for X = A, AI, AII, AIII well-understood.*
- *Procedure should extend* mutatis mutandis *to other CAZ classes.*

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Theorem

$\left(\mathrm{e}^{-\mathrm{i}tM}\right)^{*}\mathcal{E}_{\mathrm{B}}$ *(endowed with the time-transported symmetries $V(t) = \mathrm{e}^{+\mathrm{i}tM}\, V\, \mathrm{e}^{-\mathrm{i}tM}$) and $\mathcal{E}_{\mathrm{B}}$ are equivalent X-vector bundles.*

Corollary

All topological invariants of class X are left invariant by the light dynamics $\mathrm{e}^{-\mathrm{i}tM}$.

Going Beyond the Linear Periodic Case

Generalizations of

Theorem

$\left(e^{-itM}\right)^* \mathcal{E}_B$ *and* $\mathcal{E}_B$ *are equivalent X-vector bundles.*

1. **Linear, *adiabatically perturbed* PTIs**
   $\rightsquigarrow$ Pseudodifferential operators on X-vector bundles
2. ***Non-linear*, periodic PTIs**

Going Beyond the Linear Periodic Case

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   ⇝ Pseudodifferential operators on X-vector bundles
2. ***Non-linear*, periodic PTIs (Today!)**

Part 2

Non-Linear Photonic Crystals

Non-Linear Material Weights with Symmetries

Assumption (Non-linear material weights)

Suppose the non-linear material weights are of the form

$$\mathcal{W}(\Psi) = W_1 \Psi + \lambda^{q-1} W_q(\Psi)$$

where q is the degree of the non-linearity *and*

$$\big(W_q(\Psi)\big)(t,x) = \int_{\mathbb{R}^q} \mathrm{d}s' \, w_q\big(x, t\,\mathbb{1} - s'; \Psi(s'_1, x), \ldots, \Psi(s'_q, x)\big)$$

is defined in terms of $\mathbb{1} := (1, \ldots, 1)$ and the q-form w_q that satisfies:

1. $w_q(x, s; \Psi_1, \ldots, \Psi_q) \in \mathbb{C}^6$ *is symmetric under exchange of* $\{\Psi_1, \ldots, \Psi_q\}$.
2. Causality and invariance: $w_q(x, s; \Psi_1, \ldots, \Psi_q) = 0$ *for all* $x \in \mathbb{R}^3$, $\Psi_1, \ldots, \Psi_q \in \mathbb{C}^6$ *and* $s_j > 0$ *for some* $j = 1, \ldots, q$.
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Non-Linear Maxwell Equations

Non-linear Maxwell equations

Dynamical equations

$$\begin{aligned} \mathrm{i}\,\delta\,\partial_t\Psi &= \mathrm{Rot}\,\mathcal{W}(\Psi) - \mathrm{i}\mathcal{J} \\ &= \big(\mathrm{Rot}\,W_1\Psi - \mathrm{i}\mathcal{J}\big) + \lambda^{q-1}\,\mathrm{Rot}\,W_q(\Psi) \end{aligned}$$

Absence of sources

$$\mathrm{Div}\big(\mathcal{W}(\Psi)\big) = 0$$

$\lambda \ll 1$ quantifies strength of non-linearity

$\delta \ll 1$ long time limit (many oscillations over period of observation)

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Symmetries of the Non-Linear Material Weights

Definition (Symmetries of the material weights)

Let V be a discrete symmetry of Rot. Then V is a symmetry of $\mathcal{W}$ iff

$$V\mathcal{W}(\Psi) = \mathcal{W}(V\Psi).$$

Existence and Uniqueness of Solutions

Theorem (Babin, Figotin (2001))

*For small enough λ there exists $T \in \mathbb{R} \cup \{\infty\}$ such that the **non-linear** Maxwell equations have a **unique** solution*

$$(-\infty, T] \ni t \mapsto \Psi_\lambda(t/\delta)$$

*up to time T which can be expressed in terms of the solution $\Psi_0(t)$ of the **linear** Maxwell equations via the **analytic map***

$$\begin{aligned}\Psi_0 \mapsto \Psi_\lambda(t/\delta) &= \big(\mathcal{U}(\Psi_0)\big)(t/\delta) \\ &= \Psi_0(t/\delta) + \sum_{j=q}^{\infty} \lambda^{j-1} \big(U_j(\Psi_0)\big)(t/\delta).\end{aligned}$$

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Existence and Uniqueness of Solutions

Corollary (Babin, Figotin (2001))

There are recursion relations for all the U_j, and the ***first non-linear response*** *is given by*

$$\begin{aligned}\big(U_q(\Psi_0)\big)(t/\delta) &= -\mathrm{i}\int_0^{t/\delta} \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(t-s)M}\, M \,\mathrm{Rot}\,\big(W_q(\Psi_0)\big)(s) \\ &= +\mathrm{i}\int_0^{t/\delta} \mathrm{d}s \int_{\mathbb{R}^q} \mathrm{d}s'\, \mathrm{e}^{-\mathrm{i}(t-s)M}\, M\,\mathrm{Rot}\,\cdot \\ &\qquad \cdot\, w_q\big(\hat{x}, s\,\mathbb{1} - s'; \Psi_0(s'_1), \ldots, \Psi_0(s'_q)\big).\end{aligned}$$

Part 3

Candidates for Relevant States

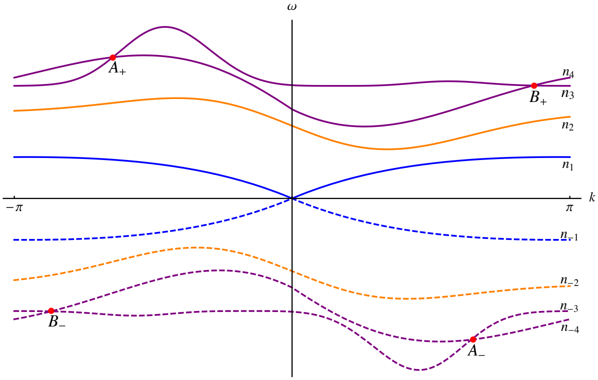
Goal

For simplicity: $\mathcal{J} = 0$ (absence of currents)

Goal

Find **initial conditions** $\{f_n(k)\}_{n\in\mathcal{I}}$ supported on a finite family of bands $\sigma_{\text{rel}}(k) = \bigcup_{n\in\mathcal{I}}\{\omega_n(k)\}$ which

1. only interact with states inside of $\sigma_{\text{rel}}(k)$,
2. are potentially topologically non-trivial and
3. have appreciable non-linear interaction.



Results by Babin & Figotin

- Arguments based on 4 works by Babin & Figotin (2001–2005)
- Uses stationary phase argument
- Focus on first non-linear response
- Arguments **depend only on frequency spectrum** (not band topology)
- Too early to consider PTIs (include discussion of C, though)

The First Non-Linear Response

The first two terms of the time-evolved coefficients

$$\big(\mathcal{F}\mathcal{U}(\Psi_0)\big)(t,k) = \sum_{n\in\mathbb{Z}\setminus\{0\}} g_n(t,k)\,\varphi_n(k)$$

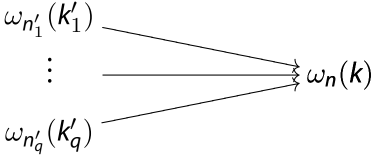
$$g_n(t,k) = e^{-it\omega_n(k)}\,f_n(k) + \lambda^{q-1} \sum_{n'\in(\mathcal{I})^3} g^{(q)}_{n,n'}(t,k) + \mathcal{O}(\lambda^q)$$

are the linearly evolved contribution and the first non-linear response.

The First Non-Linear Response

$$g^{(q)}_{n,n'}(t/\delta, k) = \frac{1}{\delta} \int_0^t \mathrm{d}s \int_{(\mathbb{T}^*)^q} \mathrm{d}k' \, \delta_{\Gamma^*}\big(k - \textstyle\sum_{j=1}^{3} k'_j\big) \, \mathrm{e}^{+\frac{\mathrm{i}}{\delta} s \, \phi_{n,n'}(k,k')} \, A_{n,n'}(k,k')$$

describes the interaction of q **incoming waves** to give **outgoing wave** with frequency $\omega_n(k)$.



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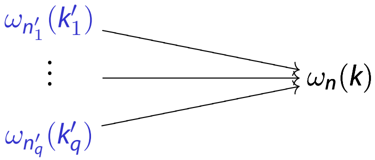

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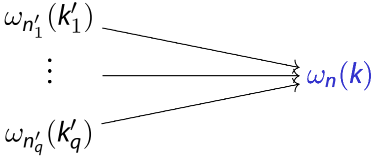

3 Matching Conditions

1. **Bloch momentum conservation** (always)

$$k = \sum_{j=1}^{q} k'_j \mod \Gamma^*$$

2. **Group velocity matching** (stationary phase $\delta \to 0$)

$$\nabla_k \omega_{n'_1}(k'_1) = \ldots = \nabla_k \omega_{n'_q}(k'_q)$$

3. **Frequency Matching** (**optional**, stationary phase $\delta \to 0$)

$$\omega_n(k) = \sum_{j=1}^{q} \omega_{n'_j}(k'_j)$$

"Generic" Existence of Non-Linearly Interacting States

What do I mean by "generic"?

Existence of $\{f_n(k)\}$ satisfying 3 matching conditions ensured either by

1. **symmetry** or
2. by **fundamental properties of the band spectrum**.

$\Rightarrow$ Stability under symmetry-preserving perturbations.

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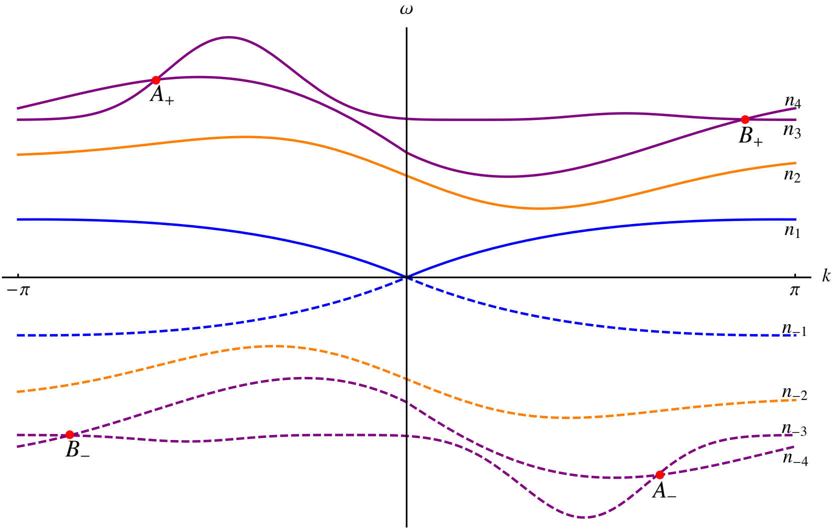

Symmetries and Non-Linearities

Symmetry	Action	$\omega_n(k) \in \sigma\big(M(k)\big) \Rightarrow$
Parity P	$P\,M(k)\,P = -M(-k)$	$\omega_{-n}(k) = -\omega_n(-k)$
C	$C\,M(k)\,C = -M(-k)$	$\omega_{-n}(k) = -\omega_n(-k)$
$J = \sigma_3 \otimes \mathrm{id}$	$J\,M(k)\,J = -M(+k)$	$\omega_{-n}(k) = -\omega_n(+k)$
$T = J\,C$	$T\,M(k)\,T = +M(-k)$	$\omega_n(k) = +\omega_n(-k)$

Symmetries and Non-Linearities

- Usually $q = 2$ (first non-linear response is quadratic)
- Presence of parity $\Rightarrow q = 3$ (medium is cubic)

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Candidates for Generic, Non-Linearly Coupling Modes

Properties	**Cubic**	**Quadratic**
Band indices	$n'_1 = n = n'_2, n'_3 = -n$	$n'_1 = n, n'_2 = \mathrm{gs}$
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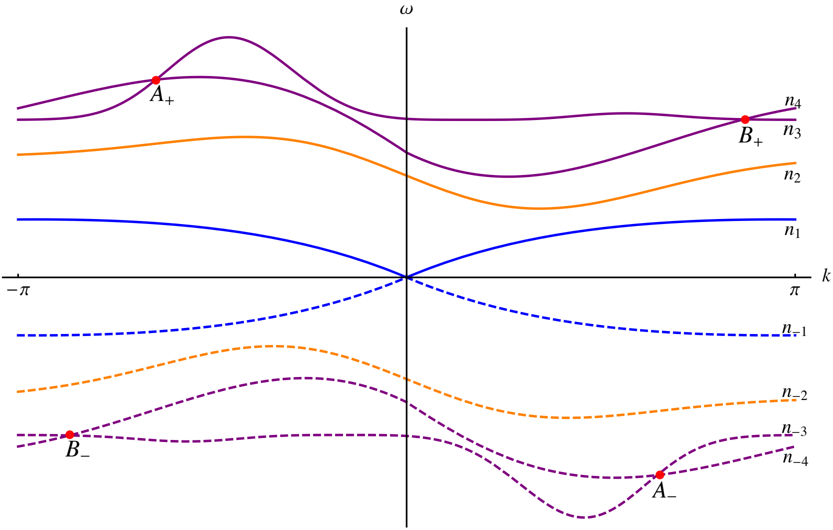

Part 4

Persistence of Topological Effects

Setting

For simplicity: cubic case ($q = 3$, parity symmetry present)

Assumption (Choice of bands)

Assume we choose the relevant bands $\sigma_{\mathrm{rel}}(k) = \bigcup_{n \in \mathcal{I}} \{\omega_n(k)\}$ so that

1. *they are separated by a **local spectral gap**,*
2. *are **symmetric** with respect to all discrete symmetries of M, and*
3. *for all $n \in \mathcal{I}$ the **only** modes which couple are of generic type.*

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The Effective Non-Linear Dynamics

Ingredient from analysis
Consider approximate dynamics

$$\big(\mathcal{U}_{\mathrm{FNLR}}(\Psi)\big)(t,k) = \sum_{n\in\mathcal{I}} \mathrm{e}^{-\mathrm{i}t\omega_n(k)} \Bigg(f_n(k) + \\ + \lambda\,\delta^2\,t\,C\,\frac{\mathrm{e}^{+\mathrm{i}\frac{\pi}{4}\operatorname{sgn}\mathrm{Hess}_{k'}\,\phi_{n,(n,n,-n)}(k,(k,k,-k))}}{\Big|\det \mathrm{Hess}_{k'}\,\phi_{n,(n,n,-n)}\big(k,\,(k,k,-k)\big)\Big|^{1/2}} \cdot \\ \cdot A_{\mathrm{eff}}\big(\{f_n\}_{n\in\mathcal{I}};k,(k,k,-k)\big)\Bigg)\,\varphi_n(k)$$

which include the first non-linear response.

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$$\mathcal{U}_{\mathsf{FNLR}}(\Psi) = U_{\mathsf{lin}}(\Psi) + \lambda\,\delta^2\,U_{\mathsf{FNLR}}(\Psi)$$

which include the first non-linear response.

Combine *Assumption (3) with additional assumption*

$$\left|\det \mathrm{Hess}_{k'}\,\phi_{n,n'}(k,k)\right| \geq C > 0$$

$\Rightarrow (t,k) \mapsto \mathcal{U}_{FNLR}(t,k)$ is continuous and maps the range of

$$P_{\mathcal{I}} := \int_{\mathbb{T}^*}^{\oplus} \mathrm{d}k \sum_{n \in \mathcal{I}} |\varphi_n(k)\rangle\langle\varphi_n(k)|$$

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Essential Ingredient

$$\mathcal{E}_{\mathrm{B}} \xleftrightarrow{\text{1-to-1}} \mathcal{E}_{S(B)} \xleftrightarrow{\mathcal{U}_{\mathrm{FLNR}}(t)} \mathcal{E}_{\widetilde{S(B)}} \xleftrightarrow{\text{1-to-1}} \mathcal{E}_{\mathrm{B}}$$

- Frame bundle $\mathcal{E}_{S(B)}$ associated to $\mathcal{E}_{\mathrm{B}}$
- $\mathcal{E}_{\widetilde{S(B)}}$ "Deformed frame bundle"
- $\mathcal{U}_{\mathrm{FNLR}}(t) = U_{\mathrm{lin}}(t) + \mathcal{O}(\lambda\,\delta^2)$ frame bundle isomorphism (changes length and orientation of vectors, remain linearly independent)

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"Non-linear dynamics do not alter topology of the Bloch bundle."

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- $\mathcal{U}_{\mathrm{FNLR}}(t)$ acts "not quite" fiber-wise evaluates at $\pm k$
  $\rightsquigarrow$ Replace $\mathcal{E}_{\mathrm{B}} \cong \mathcal{E}_+ \oplus \mathcal{E}_-$ with $\mathcal{E}_+ \oplus f^* \mathcal{E}_-$ where $f : k \mapsto -k$
- Include symmetries

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- Include symmetries

Topological Invariants

Then we immediately deduce

Theorem

All topological invariants of class X are left invariant by the ***non-linear*** *dynamics.*

Conclusion

Covered in the talk today

Part 1

Classification of *Linear* PTIs

- Schrödinger formalism of electromagnetism
  $\rightsquigarrow$ application of CAZ scheme for TIs
- Complete classification table in publication
- Ordinary material in class BDI (3 symmetries)
  $\rightsquigarrow$ different from time-reversal-invariant quantum systems!
  $\rightsquigarrow$ each symmetry can be broken individually

Part 2

Dynamics and Symmetries of Non-Linear PhCs

- For small coupling: existence and uniqueness of solutions
- Explicit expression of first non-linear response

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Part 2

Dynamics and Symmetries of Non-Linear PhCs

- For small coupling: existence and uniqueness of solutions
- Explicit expression of first non-linear response

Covered in the talk today

Part 3

Candidates for Relevant States

- Notion of non-linearly coupling modes which exist “generically”
- Cubic case: generic non-linearly coupling modes exist
- Quadratic case: generic non-linearly coupling modes exist
  $\rightsquigarrow$ outside of current theory

Part 4

Topological Invariants & Non-Linear Dynamics

- How to prove stability of topological invariance in presence of non-linearities

Thank you for your attention!

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