Analysis of ΨDOs by Combining Analytic and Algebraic Techniques

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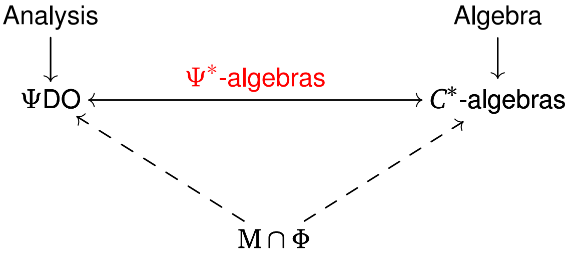
Talk based on

1. *Magnetic pseudodifferential operators with coefficients in C^*-algebras*, Publ. RIMS Kyoto Univ. **46**, pp. 755–788, 2010
2. *Magnetic twisted actions on general abelian C*-algebras*, with F. Belmonte and M. Mňtoiu, Journal of Operator Theory **69** no. 1, pp. 33–58, 2013
3. *Semiclassical Dynamics and Magnetic Weyl Calculus*, PhD thesis

Basic idea


Analysis
Algebra
Ψ^*-algebras
ΨDO
C^*-algebras
$\mathrm{M} \cap \Phi$

Ψ^*-algebras

Definition (Ψ^*-algebra Gramsch (1984))

Let $\mathfrak{C}$ be a unital C^*-algebra and $\Psi \subseteq \mathfrak{C}$ a $*$-subalgebra with unit. Then Ψ is a Ψ^*-algebra if and only if

1. Ψ can be endowed with a Fréchet topology τ_Ψ such that the embedding $\Psi \hookrightarrow \mathfrak{C}$ is continuous and
2. $\Psi \cap \mathfrak{C}^{-1} = \Psi^{-1}$ holds.

My motivation

1. Analysis of magnetic pseudodifferential operators using algebraic methods.
2. *Application to problems of solid state physics:* Conduction properties of crystalline solids
3. *Application to problems of optics:* Light conduction properties of photonic crystals
4. *Long-term goal »Adiabatic perturbation theory for random systems«:* Combination of non-commutative geometry and magnetic ΨDO techniques

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Pseudodifferential operators on $\mathbb{R}^d$

Building block operators

Weyl quantization is a functional calculus for

$$Q = \hat{x}$$
$$P = -i\nabla_x$$

Commutation relations

$$\mathrm{i}\,[Q_l, Q_j] = 0 \qquad \mathrm{i}\,[P_l, Q_j] = \delta_{lj} \qquad \mathrm{i}\,[P_l, P_j] = 0$$

Weyl quantization

For suitable functions $h : \Xi \longrightarrow \mathbb{C}$ on phase space $\Xi = \mathbb{R}^d \times \mathbb{R}^{d^*}$, $u \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$:

$$\begin{aligned}\big(\mathfrak{Op}(h)u\big)(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^{d^*}} \mathrm{d}\eta \, e^{-i(y-x)\cdot\eta} \, . \\ &\qquad \cdot h\Big(\tfrac{1}{2}(x+y), \eta\Big) \, u(y) \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathcal{F}h\Big(\tfrac{1}{2}(x+y), y-x\Big) \, u(y)\end{aligned}$$

The Moyal product

$$\mathfrak{Op}\big(f \sharp g\big) := \mathfrak{Op}(f)\,\mathfrak{Op}(g)$$

$$\big(f \sharp g\big)(x,\xi) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d \times \mathbb{R}^{d*}} \mathrm{d}y\,\mathrm{d}\eta \int_{\mathbb{R}^d \times \mathbb{R}^{d*}} \mathrm{d}z\,\mathrm{d}\zeta\; \mathrm{e}^{-i(\eta\cdot y - y\cdot\zeta)}\,. \\ \cdot f\Big(x - \tfrac{1}{2}y, \xi - \eta\Big)\, g\Big(x - \tfrac{1}{2}z, \xi - \zeta\Big)$$

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\cdot \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}z \, \mathcal{F}f\Big(y, z - \tfrac{1}{2}(x-y)\Big) \, \mathcal{F}g\Big(x-y, z + \tfrac{1}{2} y\Big)
\end{aligned}$$

Hörmander classes

Definition

The Fréchet space of Hörmander symbols of order m and type $\rho \in [0, 1]$ are defined as

$$S^m_\rho := \Big\{ f \in \mathcal{C}^\infty(\Xi) \;\Big|\; \forall a, \alpha \in \mathbb{N}_0^d \; \exists C_{a\alpha} > 0 : \\ \big| \partial_x^a \partial_\xi^\alpha f(x, \xi) \big| \leq C_{a\alpha} \, \langle \xi \rangle^{m - |\alpha| \rho} \;\; \forall (x, \xi) \in \Xi \Big\}.$$

Hörmander classes

Theorem

$S_\rho^{m_1} \sharp S_\rho^{m_2} \subseteq S_\rho^{m_1+m_2}$

Hörmander classes

Theorem (Bony)

Let $h = h^* \in S^m_\rho$ for $m \geq 0$. In case $m > 0$, we assume in addition that h is elliptic. Then for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$(h - z)^{(-1)_\sharp} \in S^{-m}_\rho$$

holds where the Moyal resolvent $(h - z)^{(-1)_\sharp}$ is the inverse of $h - z$ with respect to $\sharp$.

Ellipticity $\Rightarrow \mathfrak{Op}(h)^* = \mathfrak{Op}(h)$ with domain $H^m(\mathbb{R}^d)$

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Simplest relevant C^*-algebra

Definition

$\mathfrak{B} := \mathfrak{Op}^{-1}\Big(\mathcal{B}\big(L^2(\mathbb{R}^d)\big)\Big)$ with

1. C^*-norm $\|f\|_{\mathfrak{B}} := \big\|\mathfrak{Op}(f)\big\|_{\mathcal{B}(L^2(\mathbb{R}^d))}$,
2. product $f \sharp g := \mathfrak{Op}^{-1}\big(\mathfrak{Op}(f)\,\mathfrak{Op}(g)\big)$ and
3. involution $f^* := \mathfrak{Op}^{-1}\big(\mathfrak{Op}(f)^*\big)$.

f, g suitable: $f \sharp g$ coincides with Moyal product and f^* is pointwise complex conjugation.

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Anisotropy algebra

Definition

Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{bu}}(\mathbb{R}^d)$ be C^*-subalgebra with unit which is stable under translations.

Encodes the properties in the x variable of $h : \mathbb{R}^d \times \mathbb{R}^{d^*} \longrightarrow \mathbb{R}$.

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Anisotropy algebra

The triple $(\mathcal{A}, \mathbb{R}^d, \theta)$ forms a C^*-dynamical system where $\theta_x[\varphi] := \varphi(\cdot + x)$ is the group action of translations.

crossed product algebra

Proposition

The triple $\left(L^1(\mathbb{R}^d, \mathcal{A}), \star, {}^\star\right)$ *is a Banach-$*$ algebra where* $L^1(\mathbb{R}^d, \mathcal{A})$ *is the Banach space of the Bochner-integrable functions with*

1. *product*

$$(F \star G)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \, \theta_{-\frac{1}{2}(x-y)}\big[F(y)\big] \, \theta_{\frac{1}{2}y}\big[G(x-y)\big] \in \mathcal{A}$$

2. *and involution* $F^\star(x) := F^*(-x)$.

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crossed product algebra

Definition (*crossed product* algebra)

$\mathcal{A} \rtimes \mathbb{R}^d$ is the completion of the Banach-$*$ algebra $\big(L^1(\mathbb{R}^d, \mathcal{A}), \star, {}^{\star}\big)$ with respect to the C^*-norm

$$\|F\|_{\rtimes} := \sup\Big\{ \big\|\pi(F)\big\|_{\mathcal{B}(\mathcal{H})} \mid \pi \text{ non-degenerate representation on } \mathcal{H} \Big\}.$$

Fourier transformed *crossed product* algebra

Definition (Fourier transform $\mathcal{F}$)

$\mathcal{F}^{-1} : L^1(\mathbb{R}^d, \mathcal{A}) \longrightarrow \mathcal{C}_\infty(\mathbb{R}^{d^*}, \mathcal{A})$

$$\big(\mathcal{F}^{-1}F\big)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x \, \mathrm{e}^{+i\xi\cdot x} F(x) \in \mathcal{A}$$

Fourier transformed *crossed product* algebra

Definition

$\mathfrak{C}_{\mathcal{A}} := \mathcal{F}^{-1}(\mathcal{A} \rtimes \mathbb{R}^d)$ with

1. norm $\|f\|_{\mathfrak{C}_{\mathcal{A}}} := \|\mathcal{F}f\|_{\rtimes}$,
2. product $f \sharp g := \mathcal{F}^{-1}(\mathcal{F}f \star \mathcal{F}g)$,

$$(f \sharp g)(x,\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{e}^{+i\xi\cdot y} \cdot \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}z \, \mathcal{F}f\Big(y, z - \tfrac{1}{2}(x-y)\Big) \, \mathcal{F}g\Big(x-y, z + \tfrac{1}{2}y\Big)$$

and

3. involution $f^{\sharp} := \mathcal{F}^{-1}((\mathcal{F}f)^{\star}) = f^*$.

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Representations of $\mathfrak{C}_{\mathcal{A}}$

Theorem

1. $\mathfrak{Op}$ *is the »position representation« of* $\mathfrak{C}_{\mathcal{A}}$ *on* $L^2(\mathbb{R}^d)$.
2. $\mathfrak{F}\,\mathfrak{Op}\,\mathfrak{F}^{-1}$ *is the »momentum representation« of* $\mathfrak{C}_{\mathcal{A}}$ *on* $L^2(\mathbb{R}^{d^*})$.

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Ψ^*-property of S^0_ρ

Proposition

$S^0_\rho \hookrightarrow \mathfrak{B}$ is a Ψ^*-algebra.

Ψ^*-property of S^0_ρ

Proof.

1. $\mathfrak{Op}(S^0_\rho) \hookrightarrow \mathcal{B}(L^2(\mathbb{R}^d))$ continuous (Caldéron-Vaillancourt)
2. $S^0_\rho \sharp S^0_\rho \subseteq S^0_\rho$ (closedness under $\sharp$)
3. $S^0_\rho \cap \mathfrak{B}^{(-1)_\sharp} = (S^0_\rho)^{(-1)_\sharp}$ (corollary of the Bony criterion)

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Anisotropic Hörmander classes

Definition ($S^m_\rho(\mathcal{A})$)

$$S^m_\rho(\mathcal{A}) := \left\{ f \in S^m_\rho \;\middle|\; \forall a, \alpha \in \mathbb{N}_0^d, \xi \in \mathbb{R}^{d^*}\, x \mapsto \partial_x^a \partial_\xi^\alpha f(x,\xi) \in \mathcal{A} \right\}$$

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Theorem (L.-Măntoiu-Richard (2010))

$S^0_\rho(\mathcal{A}) \hookrightarrow \mathfrak{B}$ is a Ψ^*-algebra.

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Proof.

1. $\mathfrak{Op}\big(S^0_\rho(\mathcal{A})\big) \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous ($S^m_\rho(\mathcal{A}) \subseteq S^m_\rho$)
2. $S^0_\rho(\mathcal{A}) \sharp S^0_\rho(\mathcal{A}) \subseteq S^0_\rho(\mathcal{A})$ (L.-Măntoiu-Richard (2010), simple)
3. ???

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Ψ^*-property of $S^0_\rho(\mathcal{A})$

The crucial ingredient is the following

Theorem (Lauter (1998))

Let $\Psi \hookrightarrow \mathfrak{B}$ be a Ψ^*-algebra and $\Psi' \subset \Psi$ a closed $*$-subalgebra with unit. Then $\Psi' \hookrightarrow \mathfrak{B}$ endowed with the topology induced by Ψ is also a Ψ^*-algebra.

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Proof.

1. $\mathfrak{Op}(S^0_\rho(\mathcal{A})) \hookrightarrow \mathcal{B}(L^2(\mathbb{R}^d))$ continuous ($S^m_\rho(\mathcal{A}) \subseteq S^m_\rho$)
2. $S^0_\rho(\mathcal{A}) \sharp S^0_\rho(\mathcal{A}) \subseteq S^0_\rho(\mathcal{A})$ (L.-Măntoiu-Richard (2010), simple)
3. follows from Ψ^*-property of S^0_ρ and result by Lauter

$\square$

What have we won?

Determination of the essential spectrum of a ΨDOs

ΨDO $H = \mathfrak{Op}(h)$ associated to h whose x-behavior is encoded in $\mathcal{A}$

Analysis $\longrightarrow$ ΨDO

Algebra $\longrightarrow$ C^*-Algebren

Determination of the essential spectrum of a ΨDOs

Essential spectrum of an operator $H = H^* \in \mathcal{B}(\mathcal{H})$
$\rightsquigarrow$ *Calkin-Algebra* $\mathcal{C}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$

Analysis	Algebra
↓	↓
ΨDO	C^*-Algebren

Determination of the essential spectrum of a ΨDOs

What is $[\mathfrak{Op}(h)]_{\mathcal{K}(\mathcal{H})} \in \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$?

Analysis	Algebra
↓	↓
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How do you link
both points of views?

Determination of the essential spectrum of a ΨDO

1. Start wit $\mathfrak{Op}(h)$ so that $h \in S^0_\rho(\mathcal{A})$.

Analysis $\downarrow$ ΨDO

Algebra $\downarrow$ C^*-algebras

Determination of the essential spectrum of a ΨDO

2. Identify $S^0_\rho(\mathcal{A})$ with a subalgebra of $\mathfrak{C}_{\mathcal{A}} \subset \mathfrak{B}$.

Analysis $\downarrow$ ΨDO $\xrightarrow{\Psi^*\text{-algebras}}$ C^*-algebras $\uparrow$ Algebra

(Analysis $\to$ ΨDO; Algebra $\to$ C^*-algebras; ΨDO $\xrightarrow{\Psi^*\text{-algebras}}$ C^*-algebras)

Determination of the essential spectrum of a ΨDO

3. Analysis of the quotient
$\mathcal{A}/\mathcal{C}_\infty(\mathbb{R}^d) \cong \mathcal{C}_\infty(\mathcal{S}_\mathcal{A} \setminus \mathbb{R}^d) \cong \bigcup_{j\in\mathcal{I}} \mathcal{C}(\mathcal{Q}_{\infty,j})$

Analysis $\longrightarrow$ ΨDO

Algebra $\longrightarrow$ C^*-algebras

Determination of the essential spectrum of a ΨDO

4. $\mathcal{A} \rtimes \mathbb{R}^d / \mathcal{C}_\infty(\mathbb{R}^d) \rtimes \mathbb{R}^d \cong \big(\mathcal{A}/\mathcal{C}_\infty(\mathbb{R}^d)\big) \rtimes \mathbb{R}^d \cong \bigcup_{j \in \mathcal{I}} \mathcal{C}(\mathcal{Q}_{\infty,j}) \rtimes \mathbb{R}^d$

Analysis
↓
ΨDO

Algebra
↓
C^*-algebras

Determination of the essential spectrum of a ΨDO

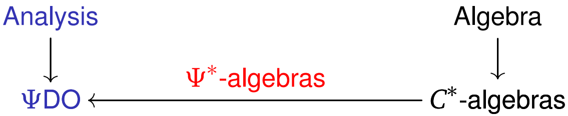
5. Morphisms $\mathfrak{C}_{\mathcal{A}} \ni f \mapsto f_{\infty,j} \in \mathfrak{C}_{\mathcal{C}(\mathcal{Q}_{\infty,j})}$

Analysis $\downarrow$ ΨDO

Algebra $\downarrow$ C^*-algebras

Determination of the essential spectrum of a ΨDO

6. The $f_{\infty,j}$ are concrete functions on $\mathbb{R}^d \times \mathbb{R}^{d^*}$
$\rightsquigarrow$ analysis of the spectra of $\mathfrak{Op}(f_{\infty,j})$.



Exact statement

Theorem (L.-Măntoiu-Richard (2010))

Let $m > 0$ and $\rho \in [0,1]$ and let $Q \subset Q_{\mathcal{A}}$ define a covering of the points at infinity $F_{\mathcal{A}}$. Moreover, assume the components of the magnetic field B are elements of $\mathcal{A}^\infty$. Then for any real-valued elliptic element h of $S^m_\rho(\mathcal{A})$, one has

$$\mathrm{spec}_{\mathrm{ess}}\left(\mathfrak{Op}^A(h)\right) = \overline{\bigcup_{\mathcal{Q}\in Q} \mathrm{spec}\left(\mathfrak{Op}^{A_\mathcal{Q}}(h)\right)}$$

where A and $A_\mathcal{Q}$ are continuous vector potentials of B and $B_\mathcal{Q}$, respectively, and $h_\mathcal{Q} \in S^m_\rho(\mathcal{A}_\mathcal{Q})$ is the image of h through $\pi_\mathcal{Q}$.

Further results

1. Magnetic fields B with components of class $\mathcal{A}^\infty$
2. Aysmptotic developments of $\sharp$ are compatible with anisotropic Hörmander classes $\rightsquigarrow$ perturbation theory
3. Extension to more general anisotropy algebras $\mathcal{A} = \mathcal{C}(\Omega)$

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Take-away message

1. The notion of Ψ^*-algebra mediates between analytic and algebraic point of view.
2. Allows for application of algebraic techniques to analytic problems (e. g. from mathematical physics).
3. Algebraic point of view provides more systematic overview of pseudodifferential calculus and simplifies arguments.

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References

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