Systematic Quantum-Wave Analogies and Applications to Topological Photonic Crystals

Using The Quantum Hall Effect for Light as a Lens

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Idea

Realizing Quantum Effects with Classical Waves

Quantum-Wave Analogies

The Quantum Hall Effect for light as a lens

The Quantum Hall Effect: the Prototypical System

$$B \neq 0 \implies$$ **time-reversal symmetry broken**

$$\sigma^{xy}_{\mathrm{bulk}}(t) \approx \frac{e^2}{h}\,\mathrm{Ch}_{\mathrm{bulk}} = \frac{e^2}{h}\,\mathrm{Ch}_{\mathrm{edge}} \approx \sigma^{xy}_{\mathrm{edge}}(t)$$

transverse conductivity = Chern #

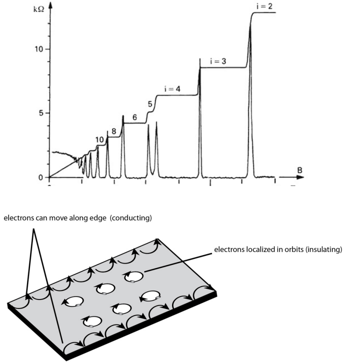
$$\mathrm{Ch}_{\mathrm{bulk/edge}} = \frac{1}{2\pi}\int_{\mathcal{B}} \mathrm{d}k\;\Omega_{\mathrm{bulk/edge}}(k) \in \mathbb{Z}$$

- Edge modes in spectral gaps
- Signed # edge channels $= \mathrm{Ch}(P_{\mathrm{Fermi}})$
- Edge modes unidirectional
- Robust against disorder

***Two* Nobel Prizes**

1985 for experiment: von Klitzing

2016 for theory: Thouless



von Klitzing et al (1980)

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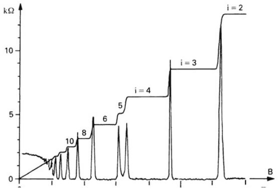
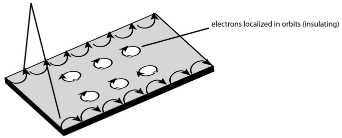




von Klitzing et al (1980)

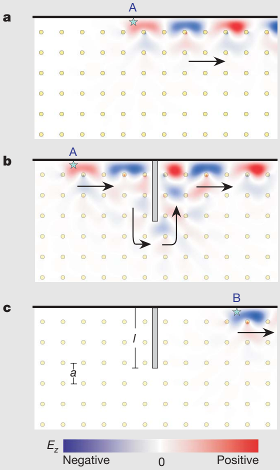
Quantum Hall Effect for Light

Predicted theoretically by Raghu & **Haldane** (2005) ...

$$\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$$

symmetry breaking

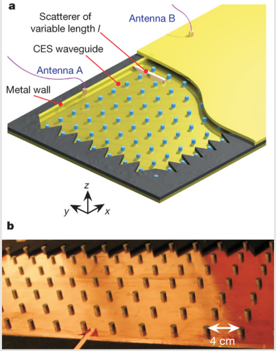
$\Longrightarrow$



Joannopoulos, Soljačić et al (2009)

Quantum Hall Effect for Light

... and realized experimentally by Joannopoulos et al (2009)



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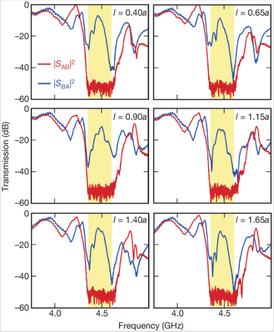


Joannopoulos, Soljačić et al (2009)

Quantum Hall Effect for Light

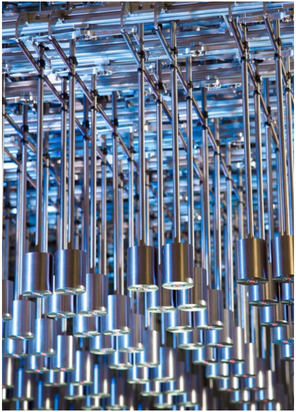
Haldane's photonic bulk-boundary correspondence

In a **two-dimensional** photonic crystals with boundary the **difference of the number of left- and right-moving boundary modes**

$$\text{signed} \sharp \text{ edge modes} = \mathrm{Ch}_{\mathrm{edge}} = \mathrm{Ch}_{\mathrm{bulk}}$$

in bulk band gaps is a **topologically protected quantity and equals the Chern number** associated to the frequency bands below the bulk band gap.

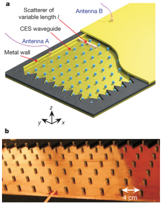
Analog of the QHE in Coupled Mechanical Oscillators



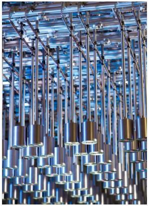
Video

Boundary mode traveling counter-clockwise.

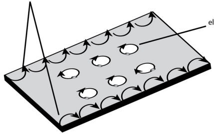
Topological Effects: Phenomenological Similarities



Light



Coupled Oscillators



Quantum

- Periodic structure ⇝ **bulk band gap**
- **Breaking** of time-reversal **symmetries**
- Unidirectional edge modes
- Robust under perturbations

Systematic Approach to Quantum-Wave Analogies

Quantum Mechanics

$$\mathrm{i}\,\partial_t \Psi = H\Psi$$
$$H = (-\mathrm{i}\nabla - A)^2 + V$$

(Schrödinger equation)

$$\longleftrightarrow \quad ?$$

Classical Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

(dynamical equations)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

- Whether and to what extent do particular quantum-wave analogies hold?
- Transfer ideas and *techniques* initially developed for quantum mechanics to classical waves.

Today's Goals

First Principles Approach to QHE for Light

1. Start with **Maxwell's equations** for media with $W \neq \overline{W}$.
   Correct equations?
2. **Schrödinger formalism** of classical electromagnetism
   First- vs. second-order formalism, restriction to $\omega \geq 0$
3. **Topological classification** of electromagnetic media
   Cartan-Altland-Zirnbauer classification for topological insulators

1. Maxwell's equations in linear, non-dispersive media
2. Schrödinger formalism of classical electromagnetism
3. Topological classification of electromagnetic media
4. Putting All The Pieces Together

Goal of This Section

Derive Maxwell's equations for gyrotropic media

Physical fields $(\mathbf{E}, \mathbf{H})$ are linear combination of complex $\pm\omega$ waves:

$$(\mathbf{E}, \mathbf{H}) = \Psi_+ + \Psi_- = 2\mathrm{Re}\, \Psi_\pm$$

Material weights

$$W_+ = W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \neq \overline{W} = W_-$$

$\Longrightarrow$ Pair of equations

$$\omega > 0 : \quad \begin{cases} W_+ \, \partial_t \Psi_+ = \begin{pmatrix} 0 & +\nabla^\times \\ -\nabla^\times & 0 \end{pmatrix} \Psi_+ \\ \mathrm{Div}\, W_+ \, \Psi_+ = 0 \end{cases}$$

$$\omega < 0 : \quad \begin{cases} W_- \, \partial_t \Psi_- = \begin{pmatrix} 0 & +\nabla^\times \\ -\nabla^\times & 0 \end{pmatrix} \Psi_- \\ \mathrm{Div}\, W_- \, \Psi_- = 0 \end{cases}$$

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Strategy

1. Start with Maxwell's equations for linear, **dispersive** media.
2. Neglect dispersion.

Crucial ingredient: *Real-valuedness of physical fields*

Fundamental Equations

Maxwell's equations in media

1. *Maxwell's equations*

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}+\nabla\times\mathbf{H}\\ -\nabla\times\mathbf{E}\end{pmatrix} - \begin{pmatrix}\mathbf{J}\\ 0\end{pmatrix} \qquad \text{(dynamical eqns.)}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix} = \begin{pmatrix}\rho\\ 0\end{pmatrix} \qquad \text{(constraint eqns.)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \mathcal{W}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge*

$$\nabla\cdot\mathbf{J} + \partial_t\rho = 0$$

Fundamental Equations

Maxwell's equations in media

1. *Maxwell's equations*

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}=\begin{pmatrix}+\nabla\times\mathbf{H}\\ -\nabla\times\mathbf{E}\end{pmatrix}$$ (dynamical eqns.)

$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$ (constraint eqns.)

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}=W\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge* $\rightsquigarrow$ neglect sources for simplicity

$$\nabla\cdot\mathbf{J}+\partial_t\rho=0$$

Constitutive Relations for Linear Media

$$(\mathbf{D}(t), \mathbf{B}(t)) = \int_{-\infty}^{t} \mathrm{d}s \, W(t-s) \, (\mathbf{E}(s), \mathbf{H}(s))$$

Assumption (Constitutive relations)

We assume that $W(t,x) = \begin{pmatrix} \varepsilon(t,x) & \chi^{EH}(t,x) \\ \chi^{HE}(t,x) & \mu(t,x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$

1. *is* **real**, $W = \overline{W}$, *and*
2. *satisfies the* causality condition $W(t) = 0$ *for all* $t < 0$.

Constitutive Relations for Linear Media

$$\big(\mathbf{D}(t), \mathbf{B}(t)\big) = \big(W * (\mathbf{E}, \mathbf{H})\big)(t)$$

Assumption (Constitutive relations)

We assume that $W(t,x) = \begin{pmatrix} \varepsilon(t,x) & \chi^{EH}(t,x) \\ \chi^{HE}(t,x) & \mu(t,x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$

1. *is* **real**, $W = \overline{W}$, *and*
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Reality Condition in Frequency Space

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \Big(\overline{\mathbf{E}(t)}\,,\, \overline{\mathbf{H}(t)}\Big)$$

$$\Longleftrightarrow$$

$$\big(\hat{\mathbf{E}}(-\omega)\,,\, \widehat{\mathbf{H}}(-\omega)\big) = \Big(\overline{\hat{\mathbf{E}}(+\omega)}\,,\, \overline{\widehat{\mathbf{H}}(+\omega)}\Big)$$

Similarly for other quantities such as $W(t)$ and $\mathbf{J}(t)$

Rewriting the Dynamical Equations

$$\frac{\partial}{\partial t} W * \Psi = -\mathrm{i}\,\mathrm{Rot}\,\Psi := -\mathrm{i} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} \Psi$$

$$\Longleftrightarrow$$

$$\mathrm{i}\,\frac{\partial}{\partial t} W * \Psi = \mathrm{Rot}\,\Psi$$

where $\Psi = (\mathbf{E}, \mathbf{H})$ is the electromagnetic field

Heuristically Neglecting Dispersion in Maxwell's Equations

$$\mathrm{i}\,\frac{\partial}{\partial t} W * \Psi(t) = \mathrm{Rot}\,\Psi(t)$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$\omega\,\widehat{W}(\omega)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

$$\Big\downarrow \approx$$

$$\pm\,\omega\,\widehat{W}(\pm\omega_0)\,\widehat{\Psi}(\pm\omega) = \mathrm{Rot}\,\widehat{\Psi}(\pm\omega)$$

$$\Big\downarrow \mathcal{F}$$

$$\widehat{W}(\pm\omega_0)\,\mathrm{i}\,\frac{\partial}{\partial t}\Psi_\pm(t) = \mathrm{Rot}\,\Psi_\pm(t)$$

1. Apply **inverse Fourier transform** in time to go from time-dependent to frequency-dependent equations.
2. Approximate material weights $\widehat{W}(\pm\omega) \approx \widehat{W}(\pm\omega_0) = W_\pm$ for frequencies $\pm\omega \approx \pm\omega_0$.
   $+\omega_0$ *and* $-\omega_0$ contributions necessary to reconstruct **real** solutions.
3. **Undo Fourier transform** to obtain dynamical equations in the **absence of dispersion**.

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Dispersion-Free Maxwell Equations for Gyrotropic Media

Real solutions linear combination of complex $\pm\omega$ waves:

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$$\left(W = \overline{W} \quad \Longleftrightarrow \quad W_- = \overline{W_+} \right)$$

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$$\left(W = \overline{W} \quad \Longleftrightarrow \quad W_- = \overline{W_+} \right)$$

Restricting to $\omega \geq 0 \neq$ Technicality!

Weights for "dual-symmetric" medium described by

$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi & \varepsilon \end{pmatrix} = \mathbb{1} \otimes \varepsilon + \sigma_1 \otimes \chi$$

where

$$\varepsilon = \begin{pmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \qquad \chi = \begin{pmatrix} 0 & +\mathrm{i}\kappa & 0 \\ -\mathrm{i}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Weights for "dual-symmetric" medium described by

$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi & \varepsilon \end{pmatrix} = \mathbb{1} \otimes \varepsilon + \sigma_1 \otimes \chi$$

commute with operator

$$U_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \sigma_1 \otimes \mathbb{1}$$

Restricting to $\omega \geq 0 \neq$ Technicality!

Weights for "dual-symmetric" medium described by

$$W = \mathbb{1} \otimes \varepsilon + \sigma_1 \otimes \chi$$

Rewrite Maxwell equations

$$\partial_t \Psi_{\uparrow/\downarrow} = \ldots$$

in **(pseudospin) eigenbasis** $\Psi_{\uparrow/\downarrow} = \psi^E \pm \psi^H$ of $U_1 = \sigma_1 \otimes \mathbb{1}$

What went wrong here?

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Restricting to $\omega \geq 0 \neq$ Technicality!

What went wrong here?

- Material weights complex!

$$W = \mathbb{1} \otimes \varepsilon + \sigma_1 \otimes \chi \neq \mathbb{1} \otimes \varepsilon - \sigma_1 \otimes \chi = \overline{W}$$

- **Maxwell equations for $\pm\omega > 0$ components different!**

$\Longrightarrow \Psi_{\uparrow/\downarrow}$ **cannot be a solution to Maxwell's equations!**

Restricting to $\omega \geq 0 \neq$ Technicality!

What went wrong here?

- Free Maxwell operator

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} = -\sigma_2 \otimes \nabla^{\times}$$

anticommutes with $U_1 = \sigma_1 \otimes \mathbb{1}$

- $\Longrightarrow U_1$ maps $\omega > 0$ states onto $\omega < 0$ states
- $\Psi_{\uparrow/\downarrow} = \psi^E \pm \psi^H$ consist of $\omega > 0$ **and** $\omega < 0$ waves

$\Longrightarrow \Psi_{\uparrow/\downarrow}$ cannot be a solution to Maxwell's equations!

Restricting to $\omega \geq 0 \neq$ Technicality!

What went wrong here?

- $\Longrightarrow \Psi_{\uparrow/\downarrow}$ violate transversality condition
- Even if (ψ^E, ψ^H) is transversal in the sense

$$\mathrm{Div}\, W(\psi^E, \psi^H) = \begin{pmatrix} \nabla \cdot (\varepsilon \psi^E + \chi \psi^H) \\ \nabla \cdot (\chi \psi^E + \varepsilon \psi^H) \end{pmatrix} = 0,$$

the eigenvectors $\Psi_{\uparrow/\downarrow}$ are *not* transversal as $\omega < 0$ obey a **different** transversality constraint $\mathrm{Div}\, \overline{W}\, \Psi_{\_} = 0$.

$\Longrightarrow \Psi_{\uparrow/\downarrow}$ **cannot be a solution to Maxwell's equations!**

Restricting to $\omega \geq 0 \neq$ Technicality!

To be continued …

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1. Start with **Maxwell's equations** for media with $W \neq \overline{W}$.
   Correct equations?
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   First- vs. second-order formalism, restriction to $\omega \geq 0$
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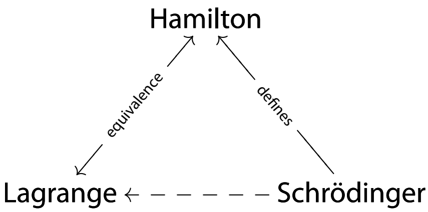
Mathematical Frameworks of Physical Theories



1. Derive **Schrödinger formalism for classical electromagnetic waves**
2. Application of Schrödinger formalism: Classify **topological photonic crystals**
3. Schrödinger and Lagrangian formalism: finding **constants of motion** in electromagnetism

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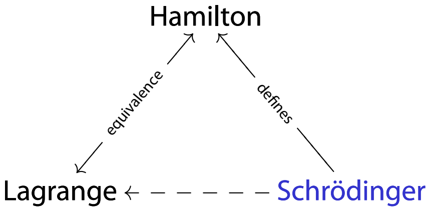

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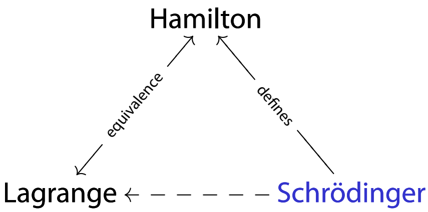

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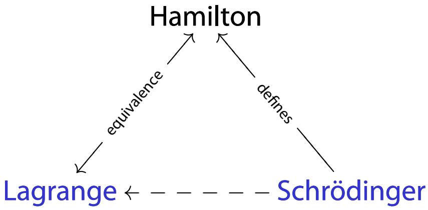

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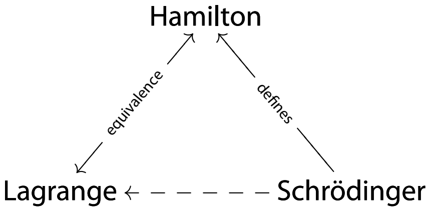

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Making Quantum-Wave Analogies Rigorous

Quantum Mechanics

$$\mathrm{i}\,\partial_t \Psi = H\Psi$$
$$H = (-\mathrm{i}\nabla - A)^2 + V$$

(Schrödinger equation)

$\longleftrightarrow$?

Classical Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

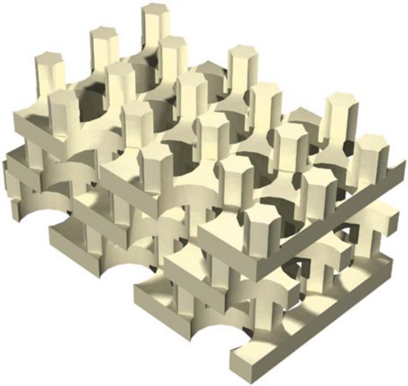
(dynamical equations)

$$\begin{pmatrix} \nabla\cdot \\ \nabla\cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

1. **States** describe the **configuration** of the system at a given time.
2. **Observables** represent experimentally **measurable** quantities.
3. **Dynamics** explain how states or observables **evolve over time.**

Assumptions on the Medium



Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. The medium is **lossless.** ($W^* = W$)
2. W describes a **positive index medium.** (eigenvalues $w_j(x)$ of $W(x)$ satisfy $0 < c \leq w_j(x) \leq C$)

Recap: States and Dynamics in Quantum Mechanics

States and Dynamics

1. A **selfadjoint Hamilton operator**, e. g.

$$H = \frac{1}{2m}\big(-\mathrm{i}\nabla - A\big)^2 + V$$
$$H = m\,\beta + \big(-\mathrm{i}\nabla - A\big) \cdot \alpha + V$$

2. A **Hilbert space** $\mathcal{H}$ and states are represented by its elements, e. g. $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with $\langle \phi, \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \phi(x) \cdot \psi(x)$.

3. **Dynamics** given by the Schrödinger equation

$$\mathrm{i}\,\partial_t \psi(t) = H \psi(t), \qquad \psi(0) = \phi$$

Recap: States and Dynamics in Quantum Mechanics

States and Dynamics

1. A **selfadjoint Hamilton operator** H
2. A **Hilbert space** $\mathcal{H}$ and states are represented by its elements.
3. **Dynamics** given by the Schrödinger equation

Properties

- $H = H^*$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}t\,H}\phi$
- $\left\lVert\psi(t)\right\rVert^2 = \lVert\psi(0)\rVert^2$ (conservation of propability)

Schrödinger Formalism of Electromagnetism

States and Dynamics

1. **"Hamilton" operator** $M_+ = W^{-1} \, \mathrm{Rot} \big|_{\omega > 0} = M_+^{*W}$ where

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}$$

2. **Hilbert space** $\mathcal{H}_+ = \big\{ \Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6) \; \big| \; \Psi \text{ is } \omega > 0 \text{ state} \big\}$
   with energy scalar product

$$\langle \Phi, \Psi \rangle_W = \int_{\mathbb{R}^3} \mathrm{d}x \, \Phi(x) \cdot W(x) \Psi(x)$$

3. **Dynamics** given by **Schrödinger equation**

$$\mathrm{i} \, \partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H}) \in \mathcal{H}_+$$

4. **Real-valuedness** of physical solutions:

$$\big(\mathbf{E}(t), \mathbf{H}(t) \big) = 2 \mathrm{Re} \, \Psi_+(t)$$

Schrödinger Formalism of Electromagnetism

States and Dynamics

1. **"Hamilton" operator** $M_+ = W^{-1} \, \mathrm{Rot} \big|_{\omega > 0} = M_+^{*W}$
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3. **Dynamics** given by **Schrödinger equation**
4. **Real-valuedness** of physical solutions

Properties

- $M_+^{*W} = M_+$
- $\Psi(t) = \mathrm{e}^{-\mathrm{i} t \, M_+} \Phi$
- $\bigl\lVert \Psi(t) \bigr\rVert_W^2 = \lVert \Psi(0) \rVert_W^2$ (conserved quantity, e. g. energy)

Obtaining the Schrödinger Formalism for EM Waves

Starting point

$$(\mathbf{E}, \mathbf{H}) = \Psi_+ + \Psi_- = 2\mathrm{Re}\,\Psi_\pm$$

$$\pm\omega > 0 : \qquad \begin{cases} W_\pm \, \mathrm{i}\partial_t \Psi_\pm = \mathrm{Rot}\, \Psi_\pm \\ \mathrm{Div}\, W_\pm \, \Psi_\pm = 0 \end{cases}$$

Strategy

1. Find a one-to-one correspondence between real-valued, physical fields $(\mathbf{E}, \mathbf{H})$ and complex waves.
2. Rewrite the dynamical Maxwell equation in Schrödinger form.
3. Verify that the solution satisfies the constraint equation.

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1. Find a one-to-one correspondence between real-valued, physical fields $(\mathbf{E}, \mathbf{H})$ and complex waves.
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Reduction to Complex Fields with $\omega > 0$

A *complex* plane wave with $\omega > 0$

$$\Psi_+(t,k,x) = \mathrm{e}^{-\mathrm{i}t\omega(k)} \, \mathrm{e}^{+\mathrm{i}k\cdot x} \, (\mathbf{E}_0, \mathbf{H}_0), \quad \omega(k) = |k| \, , \; \mathbf{E}_0, \mathbf{H}_0 \perp k,$$

defines *two* linearly independent *real* waves:

$$\begin{aligned} (\mathbf{E}_{\mathrm{Re}} \, , \mathbf{H}_{\mathrm{Re}} \,) &= \mathrm{Re} \, \Psi_+ = \cos(k \cdot x - \omega t) \, (\mathbf{E}_0, \mathbf{H}_0) \\ (\mathbf{E}_{\mathrm{Im}} \, , \mathbf{H}_{\mathrm{Im}} \,) &= \mathrm{Im} \, \Psi_+ = \sin(k \cdot x - \omega t) \, (\mathbf{E}_0, \mathbf{H}_0) \end{aligned}$$

Identification $\mathbb{R}$-VS $L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6)$ with $\mathbb{C}$-VS $\mathcal{H}_+ = \mathrm{ran} \, P_+$:

$$\alpha_{\mathrm{Re}} \, (\mathbf{E}_{\mathrm{Re}} \, , \mathbf{H}_{\mathrm{Re}} \,) + \alpha_{\mathrm{Im}} \, (\mathbf{E}_{\mathrm{Im}} \, , \mathbf{H}_{\mathrm{Im}} \,) = \mathrm{Re} \Big((\alpha_{\mathrm{Re}} - \mathrm{i}\alpha_{\mathrm{Im}} \,) \, \Psi_+ \Big)$$

Reduction to Complex Fields with $\omega > 0$

A *complex* plane wave with $\omega > 0$

$$\Psi_+(t,k,x) = \mathrm{e}^{-\mathrm{i}t\omega(k)}\,\mathrm{e}^{+\mathrm{i}k\cdot x}\,(\mathbf{E}_0,\mathbf{H}_0), \quad \omega(k) = |k|\,,\ \mathbf{E}_0,\mathbf{H}_0 \perp k,$$

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$$\alpha_{\mathrm{Re}}\,(\mathbf{E}_{\mathrm{Re}}\,,\mathbf{H}_{\mathrm{Re}}\,) + \alpha_{\mathrm{Im}}\,(\mathbf{E}_{\mathrm{Im}}\,,\mathbf{H}_{\mathrm{Im}}\,) = \mathrm{Re}\,\Big((\alpha_{\mathrm{Re}} - \mathrm{i}\alpha_{\mathrm{Im}}\,)\,\Psi_+\Big)$$

Reduction to Complex Fields with $\omega > 0$

Bloch waves with $\omega > 0$

$$\Psi_+(t,k,x) = \mathrm{e}^{-\mathrm{i}t\omega_n(k)}\,\varphi_n(k,x), \quad M_+(k)\,\varphi_n(k) = \omega_n(k)\,\varphi_n(k),$$

defines *two* linearly independent *real* waves: Still true?

$$\begin{aligned}(\mathbf{E}_{\mathrm{Re}}\,,\mathbf{H}_{\mathrm{Re}}\,) &= \mathrm{Re}\,\Psi_+\\ (\mathbf{E}_{\mathrm{Im}}\,,\mathbf{H}_{\mathrm{Im}}\,) &= \mathrm{Im}\,\Psi_+\end{aligned}$$

Identification $\mathbb{R}$-VS $L^2_{\mathrm{trans}}(\mathbb{R}^3,\mathbb{R}^6)$ with $\mathbb{C}$-VS $\mathcal{H}_+$: Still true?

$$\alpha_{\mathrm{Re}}\,(\mathbf{E}_{\mathrm{Re}}\,,\mathbf{H}_{\mathrm{Re}}\,) + \alpha_{\mathrm{Im}}\,(\mathbf{E}_{\mathrm{Im}}\,,\mathbf{H}_{\mathrm{Im}}\,) = \mathrm{Re}\,\Big(\big(\alpha_{\mathrm{Re}}\, - \mathrm{i}\alpha_{\mathrm{Im}}\,\big)\,\Psi_+\Big)$$

Reduction to Complex Fields with $\omega > 0$

Identification $\mathbb{R}$-VS $L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6)$ with $\mathbb{C}$-VS $\mathcal{H}_+$: Still true!

$$\alpha_{\mathrm{Re}} \, (\mathbf{E}_{\mathrm{Re}} , \mathbf{H}_{\mathrm{Re}}) + \alpha_{\mathrm{Im}} \, (\mathbf{E}_{\mathrm{Im}} , \mathbf{H}_{\mathrm{Im}}) = \mathrm{Re} \Big((\alpha_{\mathrm{Re}} - \mathrm{i} \alpha_{\mathrm{Im}}) \, \Psi_+ \Big)$$

Proposition (De Nittis & L. (2017))

The $\mathbb{R}$-vector space of transversal, real vector fields $L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6)$ can be canonically identified with the $\mathbb{C}$-vector space of complex positive frequency fields $\mathcal{H}_+ = P_+ \big[L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) \big]$. The vector space isomorphisms are

$$\begin{aligned} P_+ &: L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6) \longrightarrow \mathcal{H}_+ , \\ 2\mathrm{Re} &: \mathcal{H}_+ \longrightarrow L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6) . \end{aligned}$$

The *Auxiliary* Maxwell Operator

$$\begin{aligned} M_+^{\text{aux}} &= \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} \\ &= W_+^{-1} \, \mathrm{Rot} \end{aligned}$$

$M_+^{\text{aux}} = M_+^{\text{aux}*w}$ selfadjoint on *weighted* Hilbert space $L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$

$$\begin{aligned} \langle \Psi, M_+^{\text{aux}} \Phi \rangle_{W_+} &= \langle \Psi, W_+ \, W_+^{-1} \, \mathrm{Rot} \, \Phi \rangle = \langle \mathrm{Rot} \, \Psi, \Psi \rangle \\ &= \langle W_+ \, M_+^{\text{aux}} \Psi, \Phi \rangle = \langle M_+^{\text{aux}} \Psi, W_+ \, \Phi \rangle = \langle M_+^{\text{aux}} \Psi, \Phi \rangle_{W_+} \end{aligned}$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}t \, M_+^{\text{aux}}}$ **unitary**, yields **conservation of energy**

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$$M_+^{\mathrm{aux}} = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^\times \\ -\mathrm{i}\nabla^\times & 0 \end{pmatrix}$$

$$= W_+^{-1}\,\mathrm{Rot}$$

$M_+^{\mathrm{aux}} = M_+^{\mathrm{aux}\,*_W}$ selfadjoint on *weighted* Hilbert space $L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$

$$\langle \Psi, M_+^{\mathrm{aux}} \Phi \rangle_{W_+} = \langle \Psi, W_+ \, W_+^{-1} \,\mathrm{Rot}\, \Phi \rangle = \langle \mathrm{Rot}\, \Psi, \Psi \rangle$$

$$= \langle W_+ \, M_+^{\mathrm{aux}} \Psi, \Phi \rangle = \langle M_+^{\mathrm{aux}} \Psi, W_+ \, \Phi \rangle = \langle M_+^{\mathrm{aux}} \Psi, \Phi \rangle_{W_+}$$

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$$= \big\langle W_+ \, M_+^{\mathrm{aux}} \Psi , \Phi \big\rangle = \big\langle M_+^{\mathrm{aux}} \Psi , W_+ \, \Phi \big\rangle = \big\langle M_+^{\mathrm{aux}} \Psi , \Phi \big\rangle_{W_+}$$

$\Rightarrow \mathrm{e}^{-\mathrm{i} t \, M_+^{\mathrm{aux}}}$ **unitary**, yields **conservation of energy**

The *Auxiliary* Maxwell Operator

Identifying physical states
$M_+^{\mathrm{aux}} = W^{-1}\,\mathrm{Rot}$

$$M_+^{\mathrm{aux}}\,\Psi_\omega = \omega\,\Psi_\omega$$

has (pseudo) eigenfunctions also for negative frequencies $\omega < 0$!

But: the $-\omega > 0$ states of M_+^{aux} are **unphysical**
Solution
Define the spectral projection onto the physical states

$$P_+ = 1_{(0,\infty)}(M_+^{\mathrm{aux}})$$

The *Auxiliary* Maxwell Operator

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The *Auxiliary* Maxwell Operator

Identifying physical states

$M_{+}^{\mathrm{aux}} = W^{-1}\,\mathrm{Rot}$ has states for $\omega > 0$, $\omega = 0$ and $\omega < 0$

Solution

Define the spectral projection onto the physical states

$$P_{+} = 1_{(0,\infty)}(M_{+}^{\mathrm{aux}})$$

and the Hilbert space

$$\mathcal{H}_{+} = P_{+}[L^2_{W_+}(\mathbb{R}^3,\mathbb{C}^6)] = \Big\{\Psi_{+} \in L^2(\mathbb{R}^3,\mathbb{C}^6) \;\Big|\; \Psi_{+} \text{ is } \omega > 0 \text{ state}\Big\}$$

Reduction to Complex Fields with $\omega > 0$

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$$(\mathbf{E}, \mathbf{H}) = 2\mathrm{Re}\, \Psi_+ \quad \longleftrightarrow \quad \Psi_+ = P_+ (\mathbf{E}, \mathbf{H})$$

Obtaining the Schrödinger Formalism for EM Waves

Starting point

$$(\mathbf{E},\mathbf{H}) = \Psi_+ + \Psi_- = 2\mathrm{Re}\,\Psi_\pm$$

$$\pm\omega > 0 : \qquad \begin{cases} W_\pm \, \mathrm{i}\partial_t \Psi_\pm = \mathrm{Rot}\,\Psi_\pm \\ \mathrm{Div}\,W_\pm \, \Psi_\pm = 0 \end{cases}$$

Strategy

1. Find a one-to-one correspondence between real-valued, physical fields $(\mathbf{E},\mathbf{H})$ and complex waves.
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The Maxwell Operator

Restriction of auxiliary Maxwell operator to $\omega > 0$:

$$M_+ = M_+^{\mathrm{aux}} \big|_{\omega>0} = W_+^{-1} \, \mathrm{Rot} \big|_{\omega>0} = M_+^{*_{W_+}}$$

- Acts on $\mathcal{H}_+ = \left\{ \Psi_+ \in L^2(\mathbb{R}^3, \mathbb{C}^6) \;\middle|\; \Psi_+ \text{ is } \omega > 0 \text{ state} \right\}$
- Inherits selfadjointness from auxiliary Maxwell operator

Schrödinger Formalism of Maxwell's Equations

Theorem (De Nittis & L. (2017))

Real transversal states

$$(\mathbf{E},\mathbf{H}) = 2\mathrm{Re}\,\Psi_+$$

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \psi_+^E \\ \psi_+^H \end{pmatrix} = \begin{pmatrix} +\nabla \times \psi_+^E \\ -\nabla \times \psi_+^H \end{pmatrix}$$

$\longleftrightarrow$

Complex states with $\omega > 0$

$$\Psi_+ = P_+(\mathbf{E},\mathbf{H})$$

$$M_+ = W_+^{-1}\,\mathrm{Rot}\,|_{\omega>0} = M_+^{*W}$$

$$\mathrm{i}\,\partial_t \Psi_+ = M_+ \Psi_+$$

$$\mathcal{H}_+ = \bigl\{ \Psi_+ \in L^2(\mathbb{R}^3,\mathbb{C}^6) \;\big|\; \Psi_+ \text{ is } \omega > 0 \text{ state} \bigr\}$$

$$\langle \Phi, \Psi \rangle_{W_+} = \int_{\mathbb{R}^3} \mathrm{d}x\, \Phi(x) \cdot W_+(x) \Psi(x)$$

Energy scalar product

(De Nittis & L., *The Schrödinger Formalism of Electromagnetism and Other Classical Waves* (2017))

Obtaining the Schrödinger Formalism for EM Waves

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The Helmholtz Decomposition

Traditional Helmholtz decomposition

Vector fields

$$\begin{aligned} C &= C_{\perp} + C_{\parallel} \\ &= \nabla \times A + \nabla V \in L^2(\mathbb{R}^3, \mathbb{C}^3) = \mathcal{J} \oplus \mathcal{G} \end{aligned}$$

can be *uniquely* decomposed into the sum of a (transversal) divergence-free field

$$\nabla \times A \in \mathrm{ran}\, \nabla^{\times} = \ker(\nabla \cdot) = \mathcal{J}$$

and a (longitudinal) gradient field

$$\nabla V \in \mathrm{ran}\, \nabla = \ker \nabla^{\times} = \mathcal{G}$$

The Helmholtz Decomposition

Helmholtz decomposition adapted to the medium

$$\begin{aligned}\Psi &= \Psi_\perp + \Psi_\parallel \\ &\in L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) = \mathcal{J}^{\mathrm{aux}}_+ \oplus \mathcal{G}\end{aligned}$$

where the **longitudinal** gradient fields make up

$$\begin{aligned}\mathcal{G} &= \Big\{ (\nabla\varphi^E, \nabla\varphi^H) \in L^2(\mathbb{R}^3, \mathbb{C}^6) \;\big|\; \varphi^E, \varphi^H \in L^2_{\mathrm{loc}}(\mathbb{R}^3) \Big\} \\ &= \mathrm{ran}\,(\nabla, \nabla) = \ker M^{\mathrm{aux}}_+\end{aligned}$$

and the **transversal** divergence-free fields are $\langle \cdot\,, \cdot \rangle_{W_+}$-orthogonal,

$$\begin{aligned}\mathcal{J}^{\mathrm{aux}}_+ = \mathcal{G}^{\perp_{W_+}} &= \Big\{ \Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6) \;\big|\; \mathrm{Div}\, W_+ \Psi = 0 \Big\} \\ &= \ker\big(\mathrm{Div}\, W_+\big) = \mathrm{ran}\, M^{\mathrm{aux}}_+ .\end{aligned}$$

The Helmholtz Decomposition

Spectral interpretation

Longitudinal fields: eigenfunctions of $M_+^{\mathrm{aux}} \Psi_{\parallel} = 0$ to $\omega = 0$

$$\Longrightarrow \quad \mathcal{H}_+ = \Big\{ \Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6) \;\Big|\; \Psi \text{ is } \omega > 0 \text{ state} \Big\}$$
$$\subset \mathcal{G}^{\perp_{W_+}} = \mathcal{J}_+^{\mathrm{aux}} = \Big\{ \Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6) \;\Big|\; \mathrm{Div}\, W_+ \Psi = 0 \Big\}$$

$\Longrightarrow$ **States in $\mathcal{H}_+$ satisfy constraint equation.**

Obtaining the Schrödinger Formalism for EM Waves

Starting point

$$(\mathbf{E}, \mathbf{H}) = \Psi_+ + \Psi_- = 2\mathrm{Re}\,\Psi_\pm$$

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Schrödinger Formalism of Electromagnetism

States and Dynamics

1. **"Hamilton" operator** $M_+ = W^{-1} \, \mathrm{Rot} \big|_{\omega > 0}$ for $\omega > 0$
2. **Hilbert space** $\mathcal{H}_+ \subset L^2_W(\mathbb{R}^3, \mathbb{C}^6)$
3. **Dynamics** given by **Schrödinger equation**

$$\mathrm{i} \, \partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H}) \in \mathcal{H}_+$$

4. **Real-valuedness** of physical solutions:

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 2\mathrm{Re}\, \Psi_+(t)$$

Note

This also applies to **gyrotropic** materials where $W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \neq \overline{W}$.

Bloch-Floquet Theory for Photonic Crystals

Assumption (Periodic Medium)

Suppose in addition that $W_+(x)$ is periodic.

$$M_+ \cong \mathcal{F}\, M_+\, \mathcal{F}^{-1} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, M_+(k)$$

$$= \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & -(-\mathrm{i}\nabla + k)^{\times} \\ +(-\mathrm{i}\nabla + k)^{\times} & 0 \end{pmatrix}$$

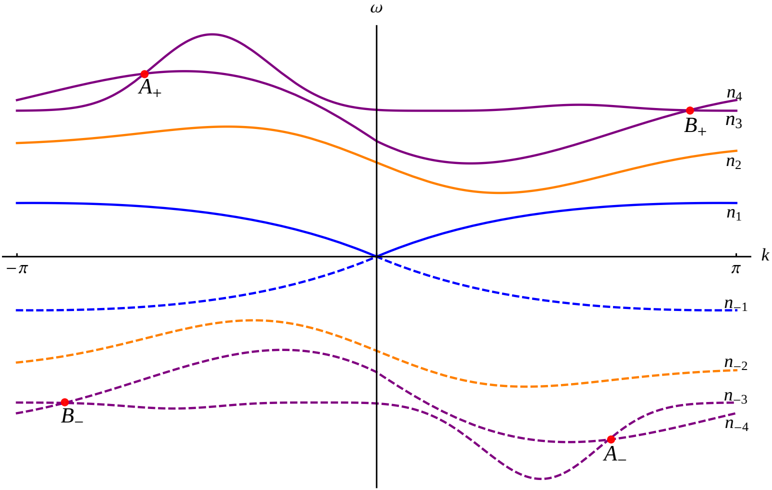
Bloch-Floquet Theory for Photonic Crystals

Physical bands

$$M_+(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
- Bloch functions $k \mapsto \varphi_n(k)$
- both **locally continuous** everywhere
- both **locally analytic** *away from band crossings*

Bloch-Floquet Theory for Photonic Crystals



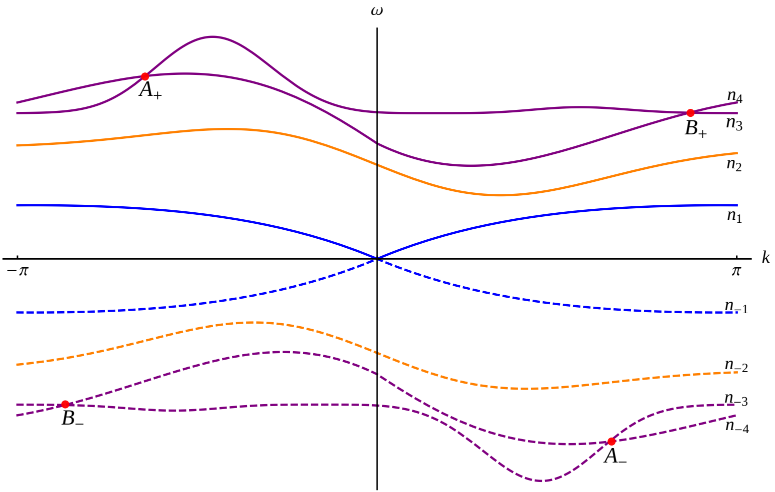
Justifying effective tight-binding models

Relevant frequency bands
$\sigma_{\mathrm{rel}}(k) = \bigcup_{n \in \mathcal{J}} \{\omega_n(k)\}$ separated by a **spectral gap** from the others.

⇝ Projection onto the relevant bands

$$P_{\mathrm{rel}}(k) = \sum_{n \in \mathcal{J}} |\varphi_n(k)\rangle\langle\varphi_n(k)|$$

Justifying effective tight-binding models



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Justifying effective tight-binding models

Idea of an effective tight-binding operator

Suppose the effective Maxwell operator M_{eff} commutes up to an error with P_{rel},

$$\bigl[M_{\mathrm{eff}} \, , \, P_{\mathrm{rel}} \bigr] = \mathcal{O}(\lambda^n),$$

and approximates the full Maxwell operators for states from the relevant bands,

$$\bigl(M_+ - M_{\mathrm{eff}} \bigr) \, P_{\mathrm{rel}} = \mathcal{O}(\lambda^n),$$

where $\lambda \ll 1$ is a perturbation parameter (that could be $\lambda = 0$).

Justifying effective tight-binding models

Idea of an effective tight-binding operator

$$\big[M_{\mathrm{eff}} \, , \, P_{\mathrm{rel}} \big] = \mathcal{O}(\lambda^n)$$
$$\big(M_+ - M_{\mathrm{eff}} \big) \, P_{\mathrm{rel}} = \mathcal{O}(\lambda^n)$$

$$\Big(\mathrm{e}^{-\mathrm{i} \frac{t}{\lambda^k} M_+} - \mathrm{e}^{-\mathrm{i} \frac{t}{\lambda^k} M_{\mathrm{eff}}} \Big) \, P_{\mathrm{rel}} =$$
$$= \int_0^t \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}s} \Big(\mathrm{e}^{-\mathrm{i} \frac{s}{\lambda^k} M_+} \, \mathrm{e}^{-\mathrm{i} \frac{(t-s)}{\lambda^k} M_{\mathrm{eff}}} \Big) \, P_{\mathrm{rel}}$$
$$= - \frac{\mathrm{i}}{\lambda^k} \int_0^t \mathrm{d}s \, \mathrm{e}^{-\mathrm{i} \frac{s}{\lambda^k} M_+} \, \underbrace{\big(M_+ - M_{\mathrm{eff}} \big) \, P_{\mathrm{rel}}}_{= \mathcal{O}(\lambda^n)} \, \mathrm{e}^{-\mathrm{i} \frac{(t-s)}{\lambda^k} M_{\mathrm{eff}}} \, P_{\mathrm{rel}}$$
$$= \mathcal{O}\big(\lambda^{n-k}\big)$$

Justifying effective tight-binding models

Idea of an effective tight-binding operator

$$\big[M_{\mathrm{eff}}\,,\,P_{\mathrm{rel}}\big] = \mathcal{O}(\lambda^n)$$
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$$\begin{aligned}
\left(\mathrm{e}^{-\mathrm{i}\frac{t}{\lambda^k}M_+} - \mathrm{e}^{-\mathrm{i}\frac{t}{\lambda^k}M_{\mathrm{eff}}}\right) P_{\mathrm{rel}} &= \\
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&= -\frac{\mathrm{i}}{\lambda^k}\int_0^t \mathrm{d}s \, \mathrm{e}^{-\mathrm{i}\frac{s}{\lambda^k}M_+} \underbrace{(M_+ - M_{\mathrm{eff}})\,P_{\mathrm{rel}}}_{=\mathcal{O}(\lambda^n)} \mathrm{e}^{-\mathrm{i}\frac{(t-s)}{\lambda^k}M_{\mathrm{eff}}}\,P_{\mathrm{rel}} \\
&= \mathcal{O}(\lambda^{n-k})
\end{aligned}$$

Justifying effective tight-binding models

Symmetry properties of effective tight-binding operator M_{eff}

$$\left(\mathrm{e}^{-\mathrm{i}\frac{t}{\lambda^k} M_+} - \mathrm{e}^{-\mathrm{i}\frac{t}{\lambda^k} M_{\mathrm{eff}}}\right) P_{\mathrm{rel}} = \mathcal{O}(\lambda^{n-k})$$

- Symmetry of $M_{\mathrm{eff}} \Longrightarrow$ symmetry of M_+
- $\Longrightarrow$ tight binding model must not possess symmetries incompatible with full Maxwell operator M_+
- Compared to M_+, the effective tight-binding operator M_{eff} may "lose" symmetries

Comparison
First- and Second-Order Formalism

First- vs. Second-Order Framework

Assume $W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$, i. e. $\chi = 0$ (no bianisotropy).

First- vs. Second-Order Framework

first order		second order
$\mathrm{i}\partial_t \binom{\psi^E}{\psi^H} = M_+ \binom{\psi^E}{\psi^H}$	$\Longleftrightarrow$	$(\partial_t^2 + M_+^2) \binom{\psi^E}{\psi^H} = 0$
$M_+ = \begin{pmatrix} 0 & +\mathrm{i}\,\varepsilon^{-1}\,\nabla^{\times} \\ -\mathrm{i}\,\mu^{-1}\,\nabla^{\times} & 0 \end{pmatrix} \Big\vert_{\omega>0}$	$\Longrightarrow$	$M_+^2 = \begin{pmatrix} \varepsilon^{-1}\,\nabla^{\times}\,\mu^{-1}\,\nabla^{\times} & 0 \\ 0 & \mu^{-1}\,\nabla^{\times}\,\varepsilon^{-1}\,\nabla^{\times} \end{pmatrix} \Big\vert_{\omega>0}$
$M_+(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$	$\Longrightarrow$	$M_+(k)^2\,\varphi_n(k) = \big(\omega_n(k)\big)^2\,\varphi_n(k)$
$\Psi(t) = \mathrm{e}^{-\mathrm{i}t\,M_+}\Psi(0)$	$\not\Longrightarrow$	$\psi^E(t) \neq \mathrm{e}^{-\mathrm{i}t\,M_E^2}\psi^E(0)$

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M_+ block-offdiagonal	$\Longrightarrow$	M_+^2 block-diagonal
$M_+(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$	$\Longrightarrow$	$M_+(k)^2\,\varphi_n(k) = \big(\omega_n(k)\big)^2\,\varphi_n(k)$
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First- vs. Second-Order Framework

second order

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Problems

- How to take $\sqrt{M_E^2}$?
- $\phi^E\big(\psi^E(0)\,,\,\psi^H(0)\big)$ depends on electric and magnetic field at time $t = 0$.
- How to distinguish between *physical* $\omega > 0$ components and *unphysical* $\omega < 0$ components?

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Compute frequency bands *starting* from

$$M_E(k)^2 \varphi_n^E(k) = \big(\lambda_n(k)\big)^2 \, \varphi_n^E(k)$$

Assumption $\lambda_n(k) \geq 0 \Longrightarrow$ yields $|\omega|$ spectrum

$\rightsquigarrow$ Sign important for dynamics!

$$0 = \big(\partial_t^2 + M_+(k)^2\big) \binom{\psi^E}{\psi^H} = \big(\partial_t + \mathrm{i}\, M_+(k)\big) \big(\partial_t - \mathrm{i}\, M_+(k)\big) \binom{\psi^E}{\psi^H}$$

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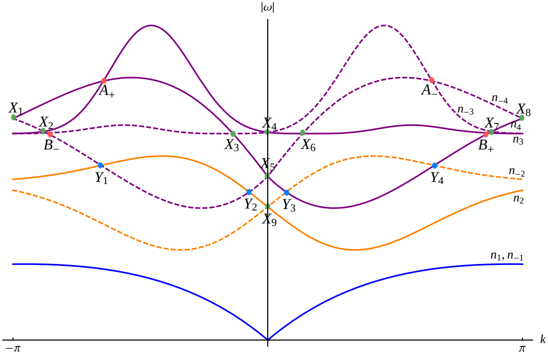
Comparison to Second-Order Formalism



1. Obtain band spectrum by solving a *second*-order equation for electric/magnetic field only, e. g. $M_+(k)_E^2\,\varphi_n^E(k) = \lambda_n(k)^2\,\varphi_n^E(k)$
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3. Use a graphene-type tight-binding model to understand light propagation for states located near intersection

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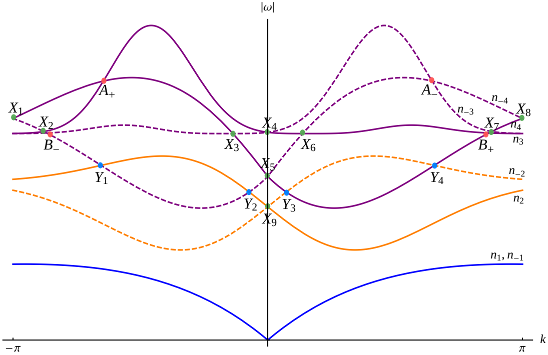

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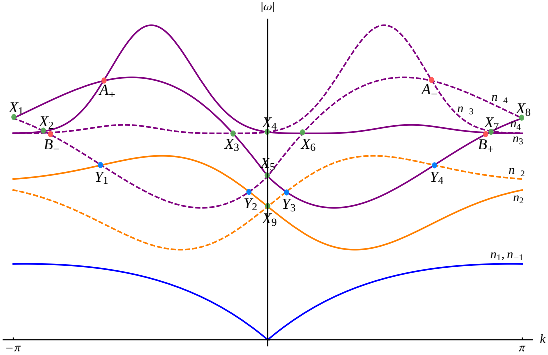

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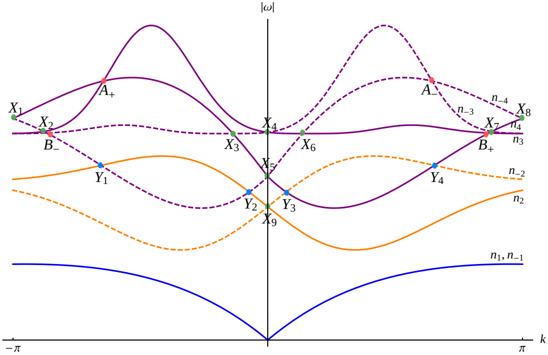

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Caution!

Procedure yields tight-binding operator M_{eff}

Problems

1. Connection of M_{eff} to dynamics?
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Today's Goals

First Principles Approach to QHE for Light

1. Start with **Maxwell's equations** for media with $W \neq \overline{W}$.
   Correct equations?
2. **Schrödinger formalism** of classical electromagnetism
   First- vs. second-order formalism, restriction to $\omega \geq 0$
3. **Topological classification** of electromagnetic media
   Cartan-Altland-Zirnbauer classification for topological insulators

Topological Phenomena as Quantum-Wave Analogies?

1. Is the **Quantum Hall Effect for Light really analogous** to the Quantum Hall Effect?
2. Are there **other topological effects**?

⇝ **Topological classification of electromagnetic media**

Material vs. Crystallographic Symmetries

Material

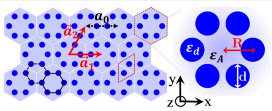
$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}$$

- Properties of and relations between ε, μ and χ
- Example:

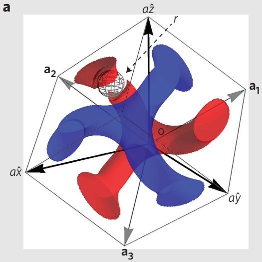
$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \neq \overline{W}, \quad \varepsilon \neq \mu$$

Only these are considered here!

Crystallographic



Wu & Hu (2015)



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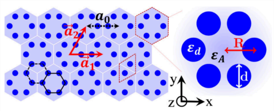

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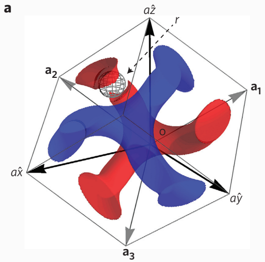
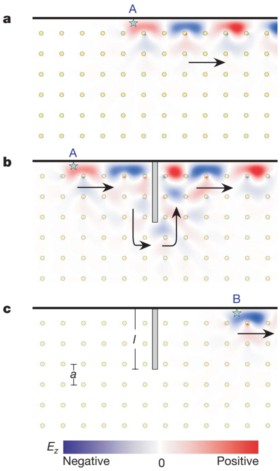


Lu et al (2013)

A Novel Class of Materials: *Photonic Topological Insulators*

$$\left.\begin{matrix}\begin{pmatrix}\overline{\varepsilon} & 0 \\ 0 & \overline{\mu}\end{pmatrix} \neq \begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix} \\ \text{symmetry breaking}\end{matrix}\right\} \Longrightarrow$$



Joannopoulos, Soljačić et al (2009)

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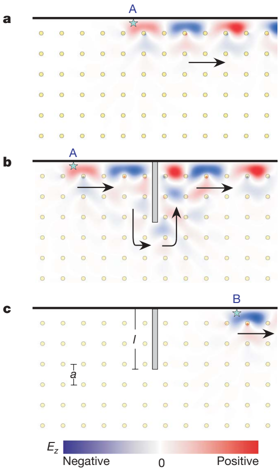

Joannopoulos, Soljačić et al (2009)

- Photonic bulk-edge correspondences

$\downarrow$

- Identify topological observables
  $O = T +$ error

$\downarrow$

- Find all topological invariants T

$\downarrow$

- Classification of PhCs by symmetries

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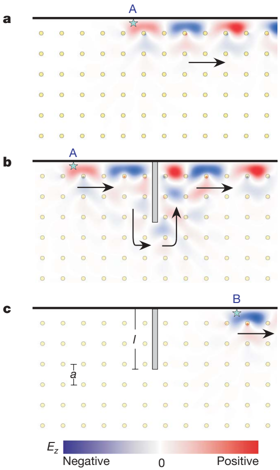

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Interjection

A Primer on Topological Insulators

Fundamental Notions

Altland–Zirnbauer Classification of Topological Insulators

The 10-fold way

1. **Topological class** $\longleftrightarrow$ Discrete symmetries of H
2. **Phases** inside each topological class $\longleftrightarrow$ Labeled by topological invariants
3. **Bulk-edge correspondences**

   physical observable $\longleftrightarrow$ topological invariant

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Topological Classes

Symmetries of H $\longleftrightarrow$ **Topological Class** of H

- **Relies on** $\mathrm{i}\partial_t \psi = H\psi$ (Schrödinger equation)
- 3 types of (pseudo) symmetries:
  U unitary/antiunitary, $U^2 = \pm\mathbb{1}$,

$U\,H(k)\,U^{-1} = +H(-k)$	time-reversal symmetry ($\pm$TR)
$U\,H(k)\,U^{-1} = -H(-k)$	particle-hole (pseudo) symmetry ($\pm$PH)
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- Relies on $\mathrm{i}\partial_t \psi = H\psi$ (Schrödinger equation)
- **3 types of (pseudo) symmetries:**
  U unitary/antiunitary, $U^2 = \pm\mathbb{1}$,

$U\,H(k)\,U^{-1} = +H(-k)$	**time-reversal symmetry ($\pm$TR)**
$U\,H(k)\,U^{-1} = -H(-k)$	particle-hole (pseudo) symmetry ($\pm$PH)
$U\,H(k)\,U^{-1} = -H(+k)$	chiral (pseudo) symmetry (χ)

- $1 + 5 + 4 = 10$ topological classes
- Physics *crucially* depends on topological class.

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Phases Inside Topological Classes

- **Inequivalent** phases inside each topological class
- *Continuous*, *symmetry-preserving* deformations of H cannot change topological phase, unless either
  – the energy gap closes (periodic case) or
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- Phases labeled by finite set of **topological invariants** (e. g. Chern numbers but also others)
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Bulk-Edge Correspondences

- Properties on the boundary can be inferred from the bulk
- Consists of 3 equalities:

$$\begin{aligned} O_{\text{bulk}}(t) &\approx T_{\text{bulk}} \\ O_{\text{edge}}(t) &\approx T_{\text{edge}} \\ T_{\text{bulk}} &= T_{\text{edge}} \end{aligned}$$

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Back to Business
Classification of Topological PhCs

No Additional Symmetries Assumption

Assumption

Apart from those below the system (i. e. the Maxwell operator M) has **no additional unitary, commuting symmetries.**

Otherwise

1. Block-decompose according to unitary, commuting symmetry.
2. Repeat until no extraneous symmetries are left.
3. Analyze each block separately with the tools used here.

Symmetries Used in Classification

Example

$$T_3 = (\sigma_3 \otimes \mathbb{1})\, C : \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \mapsto \begin{pmatrix} +\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} C \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\overline{\mathbf{E}} \\ -\overline{\mathbf{H}} \end{pmatrix}$$

Pauli matrix $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in electro-magnetic splitting

Symmetries Used in Classification

Unitary symmetries

$$U_n = \sigma_n \otimes \mathbb{1}, \quad n = 1, 2, 3$$

Antiunitary symmetries

$$T_n = (\sigma_n \otimes \mathbb{1})\, C, \quad n = 0, 1, 2, 3$$

- C is complex conjugation
- $\sigma_0 = \mathbb{1}$ the identity
- σ_1, σ_2 and σ_3 are the Pauli matrices in the $\mathbf{E}$-$\mathbf{H}$ splitting
- U_n and T_n (anti)commute with free Maxwell operator

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} = -\sigma_2 \otimes \nabla^{\times}$$

Admissible Symmetries

$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} = \sum_{n=0}^{3} \sigma_n \otimes w_n$$

where e. g. $w_0 = \frac{1}{2}(\varepsilon + \mu)$ and $w_3 = \frac{1}{2}(\varepsilon - \mu)$

Admissible Symmetries

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Symmetry $V =$	$w_0 =$	$w_1 =$	$w_2 =$	$w_3 =$	*Symmetry Type*
$T_1 = (\sigma_1 \otimes \mathbb{1})\, C$	$\mathrm{Re}\, w_0$	$\mathrm{Re}\, w_1$	$\mathrm{Re}\, w_2$	$\mathrm{i}\, \mathrm{Im}\, w_3$	+TR
$U_2 = \sigma_2 \otimes \mathbb{1}$	w_0	0	w_2	0	ordinary
$T_3 = (\sigma_3 \otimes \mathbb{1})\, C$	$\mathrm{Re}\, w_0$	$\mathrm{i}\, \mathrm{Im}\, w_1$	$\mathrm{Re}\, w_2$	$\mathrm{Re}\, w_3$	+TR

Admissibility Conditions

- Reality of $(\mathbf{E}, \mathbf{H}) \iff \omega > 0$ fields $\mapsto \omega > 0$ fields $\Longrightarrow V\, M = M\, V$
- Compatibility with energy scalar product $\Longrightarrow V\, W = W\, V$

$\Longrightarrow$ exclude *anti*commuting symmetries

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Admissibility Conditions

$\Longrightarrow$ exclude *anti*commuting symmetries

Relevance to Classification

$\Longrightarrow$ exclude **unitary, commuting** symmetries

Revisiting

Example from the Previous Section

Restricting to $\omega \geq 0 \neq$ Technicality!

Weights for "dual-symmetric" medium described by

$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi & \varepsilon \end{pmatrix} = \mathbb{1} \otimes \varepsilon + \sigma_1 \otimes \chi \neq \mathbb{1} \otimes \varepsilon - \sigma_1 \otimes \chi = \overline{W}$$

where

$$\varepsilon = \begin{pmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \qquad \chi = \begin{pmatrix} 0 & +\mathrm{i}\kappa & 0 \\ -\mathrm{i}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

commute with operator

$$U_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \sigma_1 \otimes \mathbb{1}$$

But: U_1 is **not a symmetry of the physical system!**

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But: U_1 is **not a symmetry of the physical system!**

- U_1 *anti*commutes with $\mathrm{Rot} \implies U_1 : \omega > 0 \mapsto \omega < 0$
- U_1 does *not* commute with M_+!

Back to

Topological Classification of EM Media

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix}$$

$$T_3 = (\sigma_3 \otimes \mathbb{1}) \, C$$

Gyrotropic

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \neq \begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix}$$

No symmetries

Dual-symmetric, non-gyrotr.

$$W = \begin{pmatrix} \varepsilon & -\mathrm{i}\chi \\ +\mathrm{i}\chi & \varepsilon \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} & -\mathrm{i}\,\overline{\chi} \\ +\mathrm{i}\,\overline{\chi} & \overline{\varepsilon} \end{pmatrix}$$

$$T_1 = (\sigma_1 \otimes \mathbb{1}) \, C, \; T_3 = (\sigma_3 \otimes \mathbb{1}) \, C$$

Magneto-electric

$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi & \varepsilon \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} & \overline{\chi} \\ \overline{\chi} & \overline{\varepsilon} \end{pmatrix}$$

$$T_1 = (\sigma_1 \otimes \mathbb{1}) \, C$$

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic

Class AI

Realized, e. g. dielectrics

Gyrotropic

Class A (Quantum Hall Class)

Realized, e. g. YIG for microwaves

Dual-symmetric, non-gyrotr.

Two +TR $\Longrightarrow$ $2 \times$ Class AI

Realized, e. g. vacuum and YIG

Magneto-electric

Class AI

Realized, e. g. Tellegen media

4 different topological classes of EM media

Topological Classification of EM Media

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Non-gyrotropic
Class AI
Realized, e. g. dielectrics

Gyrotropic
Class A **(Quantum Hall Class)**
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Dual-symmetric, non-gyrotr.
Two +TR $\Longrightarrow$ $2 \times$ Class AI
Realized, e. g. vacuum and YIG

Magneto-electric
Class AI
Realized, e. g. Tellegen media

Only one is topologically non-trivial in $d \leq 3$

Topological Photonic Crystals
Topology of What?

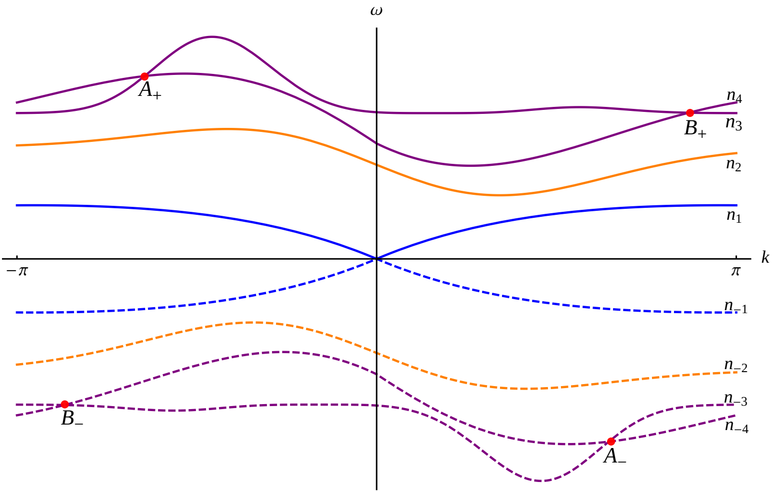
The Topology of Light States in Periodic Media

Relevant frequency bands
$\sigma_{\mathrm{rel}}(k) = \bigcup_{n \in \mathcal{J}} \{\omega_n(k)\}$ separated by a **spectral gap** from the others.

$\rightsquigarrow$ Projection onto the relevant bands

$$P_{\mathrm{rel}}(k) = \sum_{n \in \mathcal{J}} |\varphi_n(k)\rangle\langle\varphi_n(k)|$$

The Topology of Light States in Periodic Media



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The Topology of Light States in Periodic Media

$$\left.\begin{array}{c}\text{Existence of topological}\\ \text{boundary states}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Topology of the (bulk)}\\ \text{Bloch bundle}\\ \mathcal{E}_{\mathbb{B}}(P_{\mathrm{rel}}) = \big(\xi_{\mathbb{B}} \xrightarrow{\pi} \mathbb{B}\big)\end{array}\right.$$

where

$$\xi_{\mathbb{B}}(P_{\mathrm{rel}}) = \bigsqcup_{k \in \mathbb{B}} \operatorname{ran} P_{\mathrm{rel}}(k) = \bigsqcup_{k \in \mathbb{B}} \operatorname{span}\{\varphi_n(k)\}_{n \in \mathcal{J}}$$

is associated to **finitely many frequency bands*** separated by a **spectral gap** from the others.

* Not ground state bands!

The Topology of Light States in Periodic Media

"Continuous deformations" of vector bundles and equivalence

$$\mathcal{E}_{\mathbb{B}}(P_{\mathrm{rel}}) : \bigsqcup_{k \in \mathbb{B}} \operatorname{ran} P_{\mathrm{rel}}(k) \xrightarrow{\pi} \mathbb{B}$$

- Gaps must remain open
  $\Longleftrightarrow$ dimension of fiber $\operatorname{ran} P_{\mathrm{rel}}(k)$ does not change!
- Continuous deformations of M_+
  $\Longrightarrow$ continuous deformations of P_{rel}
- Continuous deformations of P_{rel}
  $\Longrightarrow$ continuous deformation of $\mathcal{E}_{\mathbb{B}}(P_{\mathrm{rel}})$

Classification of vector bundles over the torus

Find a way to characterize the Bloch vector bundle "modulo continuous deformations" $\rightsquigarrow$ This is a solved problem!

The Topology of Light States in Periodic Media

Theorem (Class A vector bundles over the torus)

For the cases of rank-m vector bundles over the d-dimensional torus listed below, the set of equivalence classes is countable and given by:

1. $d = 1, m \geq 1$: $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{S}^1) \cong \{0\}$
2. $d \geq 2, m = 1$: $\mathrm{Vec}^1_{\mathbb{C}}(\mathbb{T}^d) \cong \mathbb{Z}$
3. $d = 2, m \geq 2$: $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^2) \cong \mathbb{Z}$
4. $d = 3, m \geq 2$: $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^3) \cong \mathbb{Z}^3$
5. $d = 4, m \geq 2$: $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^2) \cong \mathbb{Z}^6 \oplus \mathbb{Z}$
6. $d \geq 5, 2m \geq d$: $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^d) \cong \mathbb{Z}^k$ for $k = \binom{d}{k}$

The Topology of Light States in Periodic Media

Theorem (Class AI vector bundles over the torus)

Suppose there exists an antiunitary operator V *with* $V^2 = +\mathbb{1}$ *and*

$$V \, P_{\mathrm{rel}}(k) \, V^{-1} = P_{\mathrm{rel}}(-k).$$

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3. $d = 4, m \geq 2$: $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^4) \cong \mathbb{Z}$
4. $d \geq 5, 2m \geq d$: $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^d) \cong \mathbb{Z}^k$

$\Longrightarrow$ All odd Chern classes vanish (first, third, etc.)

The Topology of Light States in Periodic Media

Chern classes are *computable!*

- "Abstract non-sense" tells us that vector bundles are characterized by $\mathbb{Z}$-valued Chern classes
- *But:* Given a particular set of Bloch functions, how do we compute Chern classes?
  Answer: From differential geometry! (E. g. the Berry *curvature*)

Topological Classification of Phases

Theorem (De Nittis & L. (2017))

Medium	*CAZ Class*	*Dimension* $d =$ 1	2	3	4
Gyrotropic	A	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6 \oplus \mathbb{Z}$
Non-gyrotropic	AI	0	0	0	$\mathbb{Z}$
Magneto-electric	AI	0	0	0	$\mathbb{Z}$
Dual-symmetric, non-gyrotropic	$2 \times$ AI	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

(Classification of Bloch vector bundles with symmetries.)

First and second Chern numbers

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Consequences of the Classification Result

1. Is the Quantum Hall Effect for Light really analogous to the Quantum Hall Effect?
   Answer: **Yes!** Both systems are in **Class A!**
2. Are there other topological effects?
   Answer: In $d \leq 3$ (unfortunately) no!
   (E. g. no analog of the Quantum Spin Hall Effect (class AII))
   In $d = 4$: Effects due to second Chern number or numbers?

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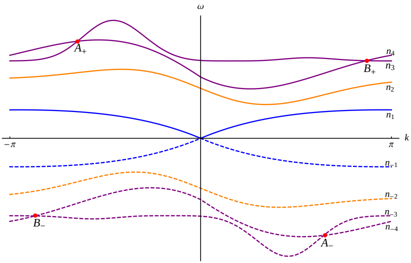
Comparison

First- and Second-Order Formalism

ω spectrum vs. $|\omega|$ spectrum

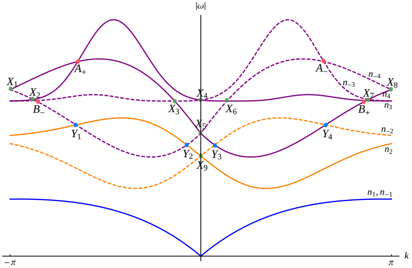
First-order formulation

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

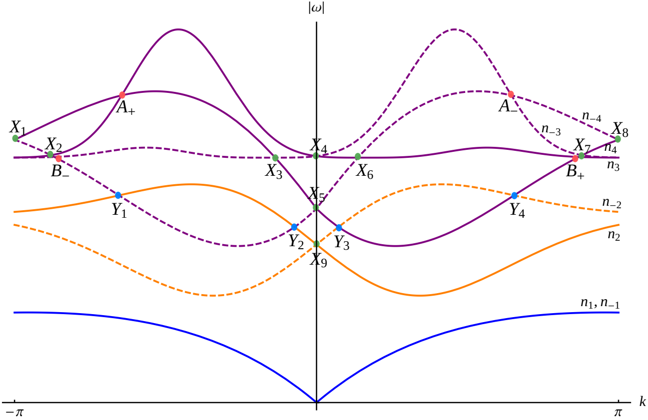


Second-order formulation

$$M(k)^2\varphi_n(k) = |\omega_n(k)|^2\,\varphi_n(k)$$



ω spectrum vs. $|\omega|$ spectrum



- Points X_j and Y_j are *artificial* band crossings
- No graphene-like physics
  ⇝ eigenfunctions well-behaved at artificial crossings

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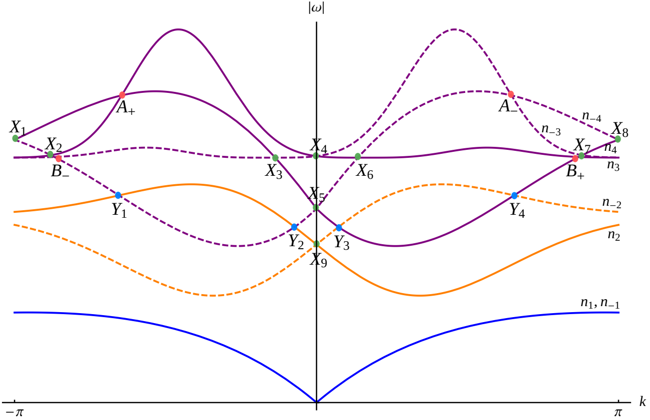

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Symmetries

Classification of (anti-)unitary U with $U^2 = \pm\mathbb{1}$ with

$$U\,M(k)^2\,U^{-1} = M(\pm k)^2$$

in Cartan-Altland-Zirnbauer scheme, e. g. if $W_+ = \overline{W_+}$

$$\left.\begin{array}{r} C\,M(k)\,C = -M(-k) \\ \Rightarrow\ C\,M(k)^2\,C = +M(-k)^2 \end{array}\right\} \quad \text{vs.} \quad \left\{\begin{array}{r} T\,M(k)\,T = +M(-k) \\ \Rightarrow\ T\,M(k)^2\,T = +M(-k)^2 \end{array}\right.$$

$\Rightarrow$ No way to distinguish PH and TR symmetry!
Ditto for chiral vs. proper symmetry

$\Rightarrow$ CAZ classification **impossible** in second-order framework!

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Proper definition of the Berry Connection

$$\mathcal{A}(k) = \mathrm{i} \left\langle \varphi_n(k), \nabla_k \varphi_n(k) \right\rangle_{W_\pm} = \mathrm{i} \left\langle \varphi_n(k), W_\pm \nabla_k \varphi_n(k) \right\rangle$$
$$= \mathrm{i} \left\langle \varphi_n^E(k), \varepsilon \nabla_k \varphi_n(k) \right\rangle + \mathrm{i} \left\langle \varphi_n^H(k), \mu \nabla_k \varphi_n^H(k) \right\rangle$$

- Berry connection sometimes computed using only $\varphi_n^E(k)$
- However: $\left\|\mathbf{E}(t)\right\|_\varepsilon^2 = \left\langle \mathbf{E}(t), \varepsilon \, \mathbf{E}(t) \right\rangle$ *not* conserved quantity!
- $\Rightarrow \mathcal{A}^E(k) = \mathrm{i} \left\langle \varphi_n^E(k), \varepsilon \nabla_k \varphi_n^E(k) \right\rangle$ *not* a connection
- Magnetic field necessary to compute Berry connection!
- Same arguments hold for φ_n^H.

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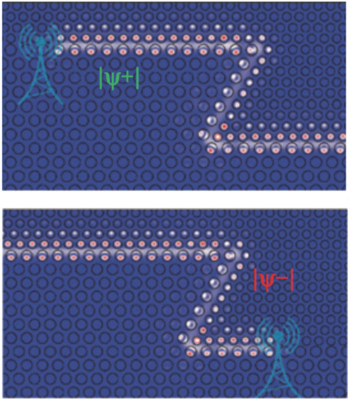
1. Maxwell's equations in linear, non-dispersive media
2. Schrödinger formalism of classical electromagnetism
3. Topological classification of electromagnetic media
4. Putting All The Pieces Together

Comparison with Other Works

Unidirectional Modes of Fixed (Pseudo)spin

- Works of Xiao Hu et al and Aleksander Khanikaev et al
- Pseudospin degree of freedom in a **time-reversal-symmetric** medium
- "Hamiltonian" aka Maxwell operator $M = \begin{pmatrix} M_\uparrow & 0 \\ 0 & M_\downarrow \end{pmatrix}$ has a block decomposition
- Topological classification *must be applied to* $M_{\uparrow/\downarrow}$
- $M_{\uparrow/\downarrow}$ of (pseudo) spin ↑/↓ **may not possess** time-reversal symmetry

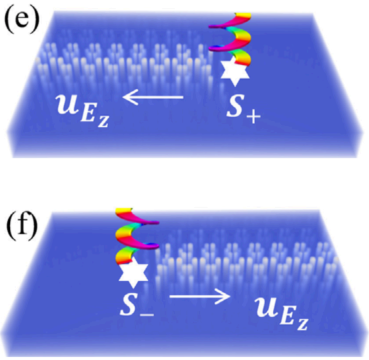


Khanikaev et al (2012)

Comparison with Other Works

Wu & Hu (2015)

- **Edge modes topological**
- Pseudospin degree of freedom in a **time-reversal-symmetric** medium
- Time-reversal symmetry $T_3 \neq T_\uparrow \oplus T_\downarrow$ **not blockdiagonal** $\Longrightarrow M_{\uparrow/\downarrow}$ class A (no symmetry)
- Chern numbers $C_\uparrow = -C_\downarrow \neq 0$ possible
- *Not* in contradiction, edge modes come in $\uparrow$ / $\downarrow$ **pairs**
- **Topologically protected** against perturbations which preserve T_3 symmetry and honeycomb structure

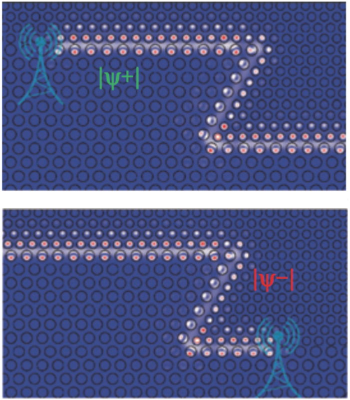


Wu & Hu (2015)

Comparison with Other Works

Khanikaev et al (2013)

- Mathematics and numerics correct
- Unfortunately, **equations unphysical**



Khanikaev et al (2012)

Thank you for your attention!