Understanding Quantum-Wave Analogies with a Focus on Spin Waves

Fusion Project in collaboration with Kōji Satō, Ryu Iguchi and Kei Yamamoto

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Advanced Institute of Materials Research, Tohoku University
2017.06.09@Tea Time

Some Relevant Wave Equations

Classical Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse Acoustic Waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \, \rho_0 \\ -\rho_0^{-1} \nabla \, \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons aka Spin Waves

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^n$

Other examples

Plasmons, magnetoplasmons, van Alfvén waves, etc.

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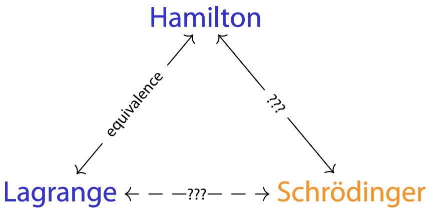
Overarching Interest

Establish Quantum-Wave Analogies

Three Frameworks in Which to Study Physical Systems



- Seemingly clear and insurmountable difference between **classical** and **quantum** worlds
- Difference in physical theories $\overset{?}{\Longrightarrow}$ Incompatibility of mathematical frameworks ⇝ **False**
- Hamiltonian and Lagrangian mechanics equivalent (subject to mathematical conditions)
- Schrödinger equation gives rise to linear Hamiltonian equations (Marsden & Ratiu, Corollary 2.5.2 & Proposition 2.6.3)
- **Koopman formalism:** Hamiltonian systems can be expressed as **linear** (!) Liouville equation
- **Case-by-case basis:** Schrödinger formalism can be established (e. g. linear electromagnetism in media, transverse acoustic waves)

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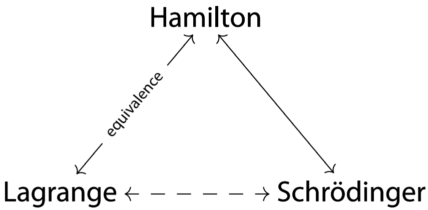

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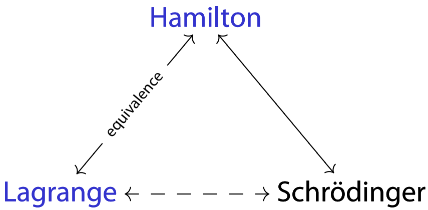

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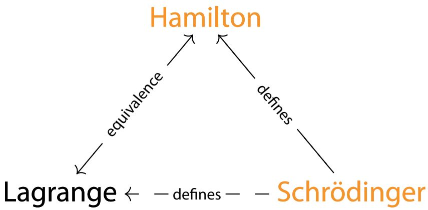

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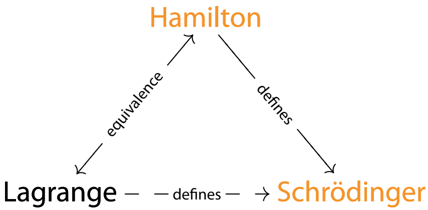

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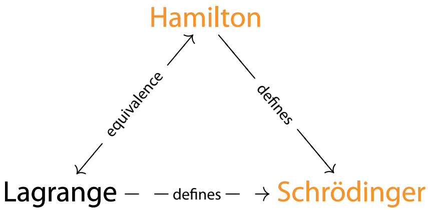

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Quantum–Wave Analogy of Interest

Topological Phenomena

The Quantum Hall Effect: the Prototypical System

physical observable $\longleftrightarrow$ *abstract mathematics*

Quantum Hall Effect

$$\sigma^{xy}_{\text{bulk}}(t) \approx \frac{e^2}{h}\,\text{Ch}_{\text{bulk}} = \frac{e^2}{h}\,\text{Ch}_{\text{edge}} \approx \sigma^{xy}_{\text{edge}}(t)$$

transverse conductivity = Chern #

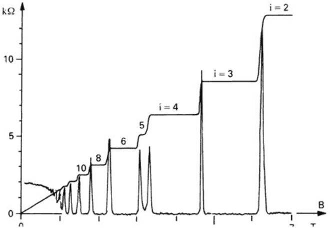
$$\text{Ch}_{\text{bulk/edge}} = \frac{1}{2\pi}\int_{\mathcal{B}} \mathrm{d}k\,\Omega_{\text{bulk/edge}}(k) \in \mathbb{Z}$$

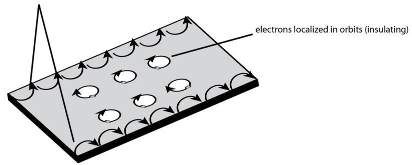
- Edge modes in spectral gaps
- Signed # edge channels $= \text{Ch}(P_{\text{Fermi}})$
- Edge modes unidirectional
- Robust against disorder

***Two* Nobel Prizes**

1980 for experiment: von Klitzing

2016 for theory: Thouless





von Klitzing et al (1980)

A Novel Class of Materials: *Topological Photonic Crystals*

Predicted theoretically by Raghu & Haldane (2005) ...

$$\left.\begin{matrix} \begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \\ \text{symmetry breaking} \end{matrix}\right\} \Longrightarrow \mathrm{Ch} = \frac{1}{2\pi} \int_{\mathcal{B}} \mathrm{d}k \, \Omega(k) \neq 0 \in \mathbb{Z}$$

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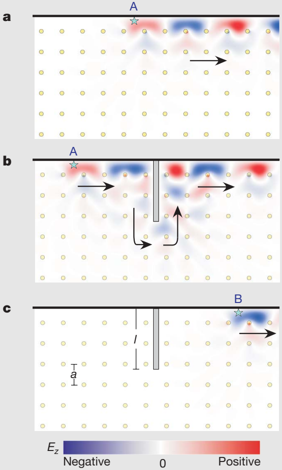
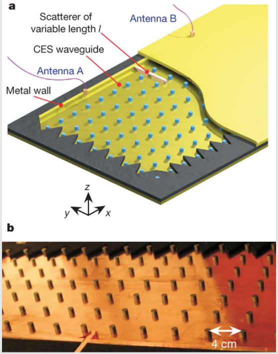


Joannopoulos, Soljačić et al (2009)

A Novel Class of Materials: *Topological Photonic Crystals*

... and realized experimentally by Joannopoulos et al (2009)



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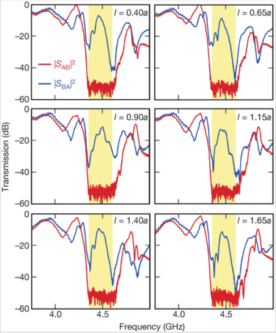
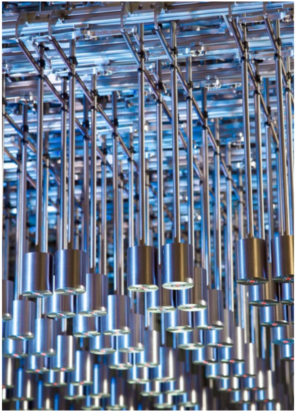


Joannopoulos, Soljačić et al (2009)

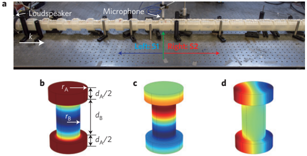
Topological Insulators for Other Waves: Experiments

Mechanical



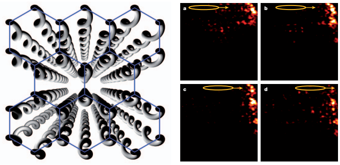
Süsstrunk & Huber (2015)

Acoustic



Xiao, Ma et al (2015)

Periodic Waveguide Arrays

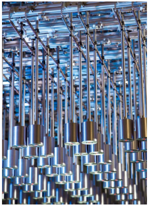


Rechtsman, Szameit et al (2013)

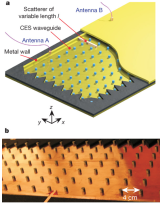
Claim: Three Experiments are

Different Manifestations of Same Underlying Physical Principles.

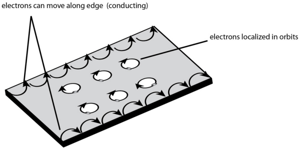
Phenomenological Similarities



Coupled Oscillators



Light



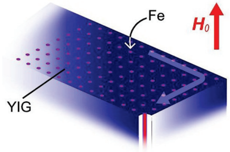
Quantum

- Periodic structure
- Breaking of time-reversal symmetries
- Boundary modes
- Robust under perturbations

What About Spin Waves?

$$\text{Ch} = \frac{1}{2\pi} \int_{\mathcal{B}} \mathrm{d}k \, \Omega(k) \overset{?!}{\in} \mathbb{Z} \qquad \Longrightarrow$$

Magnonic Crystals



Shindou, Matsumoto et al (2013)

$\rightsquigarrow$ So far not observed experimentally.

The Problem of Defining Chern Numbers for Spin Systems

	QM & EM	**Spin Waves**
Space	Hilbert space	Krein space
Inner Product	scalar product $\langle \phi, \psi \rangle$	indeterminate inner product $\langle \phi, \psi \rangle_{\sigma_3} = \langle \phi, \sigma_3 \psi \rangle$
Dynamical Equation	$\mathrm{i}\partial_t \psi(t) = H\psi(t)$	$\mathrm{i}\partial_t \psi(t) = \sigma_3 \, H\psi(t)$
Hamiltonian	selfadjoint	Krein-selfadjoint

The Problem of Defining Chern Numbers for Spin Systems

	QM & EM	**Spin Waves**
Space	Hilbert space with $\langle \phi, \psi \rangle$	Krein space with $\langle \phi, \psi \rangle_{\sigma_3} = \langle \phi, \sigma_3 \psi \rangle$
Berry "Connection"	$\mathcal{A} = \mathrm{i} \langle \varphi_n, \nabla_k \varphi_n \rangle$	$\mathcal{A} = \mathrm{i} \langle \varphi_n, \nabla_k \varphi_n \rangle_{\sigma_3}$
Berry "Curvature"	$\Omega = \partial_{k_1} \mathcal{A}_2 - \partial_{k_2} \mathcal{A}_1$	$\Omega = \partial_{k_1} \mathcal{A}_2 - \partial_{k_2} \mathcal{A}_1$
"Chern Number"	$\frac{1}{2\pi} \int_{\mathcal{B}} \mathrm{d}k \, \Omega(k) \in \mathbb{Z}$	$\frac{1}{2\pi} \int_{\mathcal{B}} \mathrm{d}k \, \Omega(k) \in \mathbb{Z}$?

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Interlude

The Schrödinger Formalism of Electromagnetism in Media

Quantum–Wave Analogies: Electromagnetism

Quantum Mechanics

$$\mathsf{i}\,\partial_t \Psi = H\Psi$$
$$H = (-\mathsf{i}\nabla - A)^2 + V$$

(Schrödinger equation)

$\longleftarrow ? \longrightarrow$

Classical Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

(dynamical equations)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

Schrödinger Formalism of Electromagnetism

$$\left.\begin{matrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} \\ \text{dynamical Maxwell equations} \end{matrix}\right\} \iff \left\{\begin{matrix} \mathrm{i}\,\partial_t \Psi = M\Psi \\ \text{“Schrödinger-type equation”} \end{matrix}\right.$$

$$\Psi(t) = \big(\mathbf{E}(t), \mathbf{H}(t)\big) \in \mathcal{H} = \Big\{ \Psi \in L^2_W(\mathbb{R}^3, \mathbb{C}^6) \;\big|\; \Psi \text{ transversal} \Big\}$$

$$M = \underbrace{\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1}}_{=W^{-1}} \underbrace{\begin{pmatrix} 0 & +(-\mathrm{i}\nabla)^{\times} \\ -(-\mathrm{i}\nabla)^{\times} & 0 \end{pmatrix}}_{=D} = M^{*_W}$$

$$\left.\begin{matrix} \text{Maxwell equations} \\ \iff \\ \text{Maxwell operator } M = M^{*_W} \end{matrix}\right\} \implies$$

Adaptation of **techniques from quantum mechanics** to electromagnetism

Schrödinger Formalism for Classical Waves

States and Dynamics

1. *"Hamilton" operator* $M = W\,D$ where
   - $W = W^*$, $0 < c\,\mathbb{1} \leq W \leq C\,\mathbb{1}$
     (positive, bounded, bounded inverse)
   - $D = D^*$ (potentially unbounded)
2. **Complex (!)** weighted *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$ where

$$\langle \Phi, \Psi \rangle_W = \langle \Phi, W \Psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \Phi(x) \cdot W(x) \Psi(x)$$

3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\,\partial_t \Psi(t) = M \Psi(t), \qquad \Psi(0) = \Phi$$

4. **Real-valuedness** of physical solutions

Schrödinger Formalism for Classical Waves

States and Dynamics

1. *"Hamilton" operator* $M = W\,D$ with **product structure**
2. **Complex (!)** weighted *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$
3. Dynamics given by *Schrödinger equation*
4. **Real-valuedness** of physical solutions

Properties

- $M^{*_W} = M$
- $\Psi(t) = \mathrm{e}^{-\mathrm{i}t\,M}\Phi$
- $\big\|\Psi(t)\big\|^2_W = \|\Phi\|^2_W$ (conserved quantity, here field energy)
- $\mathrm{Re}\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}$ where $\mathrm{Re} = \frac{1}{2}(\mathbb{1} + C)$
  (existence of real solutions)

Doubling of Degrees of Freedom

One of the **tenets of electromagnetism**:

E and **H** are **real** vector fields.

$\Longrightarrow$ Replacing $L^2_{W,\perp}(\mathbb{R}^3, \mathbb{R}^6) \rightsquigarrow L^2_{W,\perp}(\mathbb{R}^3, \mathbb{C}^6)$
doubles the degrees of freedom!

On the other hand, if we want to apply the theory of selfadjoint operators we need to work with **complex** Hilbert spaces!

Restriction to Complex Fields with $\omega > 0$

$C\,M\,C = -M \implies C\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,C$ implies

$$\mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0, \mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,(\mathrm{Re}\,\Psi_{\pm}) = \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}t\,M}\Psi_{\pm}\right)$$

where Re $= \frac{1}{2}(\mathbb{1} + C)$ is the real part operator and

$$\Psi_{+} = 1_{\{\omega>0\}}(M)\,(\mathbf{E}_0, \mathbf{H}_0) = P_{+}(\mathbf{E}_0, \mathbf{H}_0)$$
$$\Psi_{-} = 1_{\{\omega<0\}}(M)\,(\mathbf{E}_0, \mathbf{H}_0) = P_{-}(\mathbf{E}_0, \mathbf{H}_0) = C\Psi_{+}$$

are the positive and negative frequency contributions

Restriction to Complex Fields with $\omega > 0$

$$C\,M\,C = -M \quad\Longrightarrow\quad C\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,C \text{ implies}$$

$$\mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0, \mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,(\mathrm{Re}\,\Psi_{\pm}) = \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}t\,M}\Psi_{\pm}\right)$$

$$\mathrm{Re} = P_{+}^{-1} \quad\Longrightarrow\quad \text{Study } M_{+} := M|_{\mathrm{ran}\,P_{+}}$$

$$\left.\begin{array}{c} \textbf{Real}\text{ transversal states} \\ (\mathbf{E}, \mathbf{H}) = \mathrm{Re}\,\Psi_{+} \\ \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \textbf{Complex}\text{ states with } \omega > 0 \\ \Psi_{+} = P_{+}(\mathbf{E}, \mathbf{H}) \\ \mathrm{i}\,\partial_t \Psi_{+} = M_{+} \Psi_{+} \end{array}\right.$$

Fundamental Constituents

Reduced Description

1. *"Hamilton" operator* $M_+ = W^{-1} D\big|_{\text{ran } P_+}$
2. *Hilbert space* $\mathcal{H}_+ = \text{ran } P_+ \subset L^2_W(\mathbb{R}^3, \mathbb{C}^6)$
3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\, \partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H}) \in \text{ran } P_+$$

4. *Real-valuedness* of physical solutions:

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \text{Re}\, \Psi_+(t)$$

Note

This also applies to **gyrotropic** materials where $W \neq \overline{W}$.

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Back to
Magnons

Fundamental Constituents

Reduced Description

1. *"Hamilton" operator* $M_+ = \sigma_3 \, H \big|_{\omega > 0}$
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3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\, \partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+ \mathbf{S} \in \text{ran}\, P_+$$

4. *Real-valuedness* of physical solutions:

$$\mathbf{S}(t) = \text{Re}\, \Psi_+(t)$$

Crucial Difference Between EM and Spin Waves

Classical Electromagnetism

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} > 0$$

$\longleftrightarrow$

Spin Waves

$$W = \sigma_3 \not> 0$$

$M = \sigma_3 \, H$ is **not selfadjoint (hermitian)**

$$\Longrightarrow$$

Definition of $P_+ = 1_{(0,\infty)}(\sigma_3 \, H)$ (restriction to $\omega > 0$)?

Fortuitous Coincidence: Transf. to Selfadj. Operator

In this specific case: $M = \sigma_3 H$ can be diagonalized via a **Krein**-unitary

Definition (Krein unitary)

$U : \mathcal{H} \longrightarrow \mathcal{H}$ invertible with

$$\langle U\phi, U\psi\rangle_{\sigma_3} = \langle U\phi, \sigma_3 U\psi\rangle = \langle \phi, \psi\rangle_{\sigma_3} .$$

Equivalently: $U^\sharp = \sigma_3 U^* \sigma_3 = U^{-1}$

Fortuitous Coincidence: Transf. to Selfadj. Operator

In this specific case: $M = \sigma_3 \, H$ can be diagonalized via a **Krein**-unitary

$$U(k)\,M(k)\,U(k)^{-1} = \sigma_3 \begin{pmatrix} h(k) & 0 \\ 0 & h(-k) \end{pmatrix} = \begin{pmatrix} h(k) & 0 \\ 0 & -h(-k) \end{pmatrix}$$

where $h(k) > 0$

Define projection onto $\omega > 0$ states via

$$P_+(k) = U(k)^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} U(k).$$

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Claim

On ran $P_+(k)$ the weighted inner product *is* a scalar product.

Fortuitous Coincidence: Transf. to Selfadj. Operator

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On ran $P_+(k)$ the weighted inner product *is* a scalar product.

Let $\phi(k), \psi(k) \in$ ran $P_+(k)$:

$$\begin{aligned}\big\langle \phi(k), \psi(k) \big\rangle_{\sigma_3} &= \big\langle P_+(k)\phi(k), \sigma_3 \psi(k) \big\rangle \\ &= \Big\langle U(k)^{-1} \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{smallmatrix}\right) U(k)\phi(k), \sigma_3 \psi(k) \Big\rangle \\ &= \Big\langle U(k)\phi(k), \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{smallmatrix}\right) \sigma_3 U(k)\psi(k) \Big\rangle \\ &= \Big\langle U(k)\phi(k), \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{smallmatrix}\right) U(k)\psi(k) \Big\rangle \geq 0\end{aligned}$$

Strictness ($\neq 0$ for $\phi(k), \psi(k) \neq 0$) follows from $\mathbb{1} = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{smallmatrix}\right)$

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Fundamental Constituents

Reduced Description **Consistent** (even though $\sigma_3 \not> 0$)

1. *"Hamilton" operator* $M_+ = \sigma_3 H\big|_{\omega>0}$
2. *Hilbert space* $\mathcal{H}_+ = \text{ran } P_+$
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The Problem of Defining Chern Numbers for Spin Systems

	QM & EM	**Spin Waves**
Space	Hilbert space with $\langle \phi, \psi \rangle$	Krein space with $\langle \phi, \psi \rangle_{\sigma_3} = \langle \phi, \sigma_3 \psi \rangle$
Berry "Connection"	$\mathcal{A} = \mathrm{i} \, \langle \varphi_n , \nabla_k \varphi_n \rangle$	$\mathcal{A} = \mathrm{i} \, \langle \varphi_n , \nabla_k \varphi_n \rangle_{\sigma_3}$
Berry "Curvature"	$\Omega = \partial_{k_1} \mathcal{A}_2 - \partial_{k_2} \mathcal{A}_1$	$\Omega = \partial_{k_1} \mathcal{A}_2 - \partial_{k_2} \mathcal{A}_1$
"Chern Number"	$\frac{1}{2\pi} \int_{\mathcal{B}} \mathrm{d}k \, \Omega(k) \in \mathbb{Z}$	$\frac{1}{2\pi} \int_{\mathcal{B}} \mathrm{d}k \, \Omega(k) \in \mathbb{Z}$ **Yes!**

Thank you very much