New Phenomena in Classical Waves

Taking Inspiration from Quantum Mechanics

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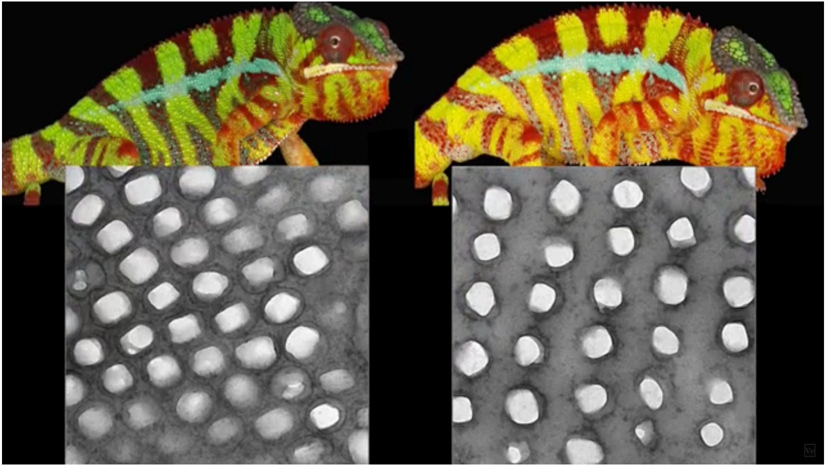
Not a Joke

What do a chameleon and a semiconductor have in common?

Video

How a chameleon changes color.

Periodic Light Conductors



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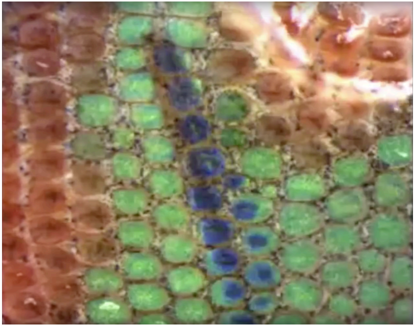


Not a Joke

What do a chameleon and a semiconductor have in common?

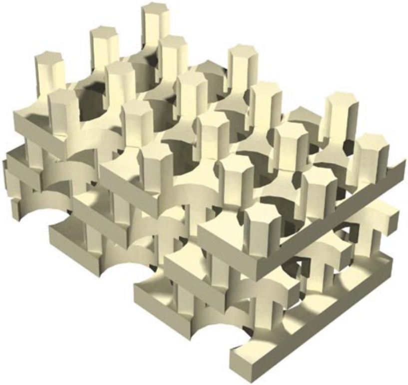
Periodicity and a band gap.

VOLUME 58, NUMBER 20 PHYSICAL REVIEW LETTERS 18 MAY 1987

Inhibited Spontaneous Emission in Solid-State Physics and Electronics

Eli Yablonovitch

Maxwell's Equations for Linear Media



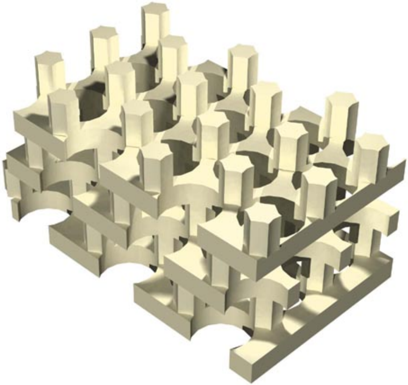
Johnson & Joannopoulos (2004)

Material weights

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

1. 6×6 matrix-valued function
2. Electric permittivity ε
3. Magnetic permeability μ
4. Phenomenologically describe properties of the medium

Maxwell's Equations for Linear Media



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Outline of Strategy

1. Reformulate Maxwell equations as a **Schrödinger equation**,

$$\mathrm{i}\partial_t \Psi(t) = M \Psi(t), \qquad \Psi(0) = \Phi \in \mathcal{H},$$

where the **Maxwell operator** $M = M^*$ is selfadjoint (hermitian) on a *complex* Hilbert space $\mathcal{H}$.

2. Use Bloch-Floquet theory to **exploit periodicity**:

$$\begin{aligned} M &\cong \int_{\mathcal{B}}^{\oplus} \mathrm{d}k \, M(k) \\ &= \int_{\mathcal{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 0 & -(-\mathrm{i}\nabla + k)^{\times} \\ +(-\mathrm{i}\nabla + k)^{\times} & 0 \end{pmatrix} \end{aligned}$$

Obtain Bloch functions and frequency bands from

$$M(k) \, \varphi_n(k) = \omega_n(k) \, \varphi_n(k)$$

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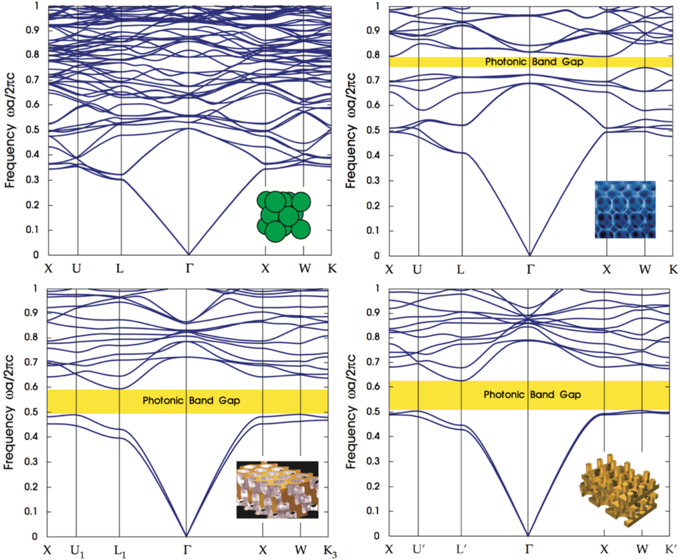

3d, taken from *Photonic Crystals – Molding the Flow of Light*

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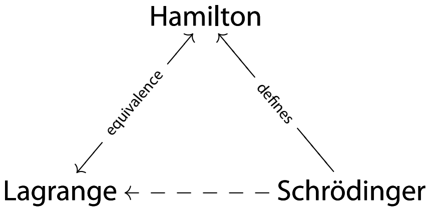
2. Use Bloch-Floquet theory to **exploit periodicity** and obtain **frequency band picture**

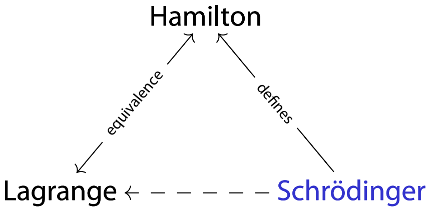
Details will come later in the talk!

Today's Goals



1. Derive **Schrödinger formalism for classical electromagnetic waves**
2. Application of Schrödinger formalism: Classify **topological photonic crystals**
3. Comparing Schrödinger and Lagrangian formalism: finding **constants of motion** in electromagnetism

Today's Goals



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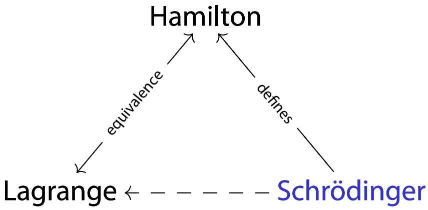

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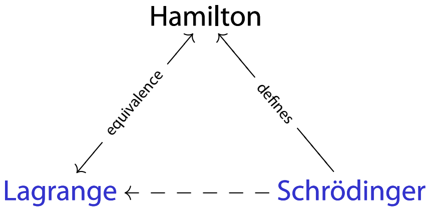

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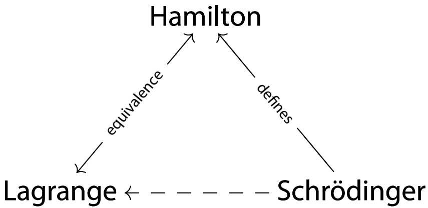

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Making Quantum-Wave Analogies Rigorous

Quantum Mechanics

$$\mathrm{i}\,\partial_t \Psi = H \Psi$$
$$H = (-\mathrm{i}\nabla - A)^2 + V$$

(Schrödinger equation)

$$\longleftarrow ? \longrightarrow$$

Classical Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

(dynamical equations)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

- Find a *systematic* scheme
- Avoid heuristics
- Take differences between theories properly into account

Basic Constituents of Physical Theories

1. States describe the **configuration** of the system at a given time.
2. Observables represent experimentally **measurable** quantities.
3. Dynamics explain how states or observables **evolve over time**.

Recap: States and Dynamics in Quantum Mechanics

States and Dynamics

1. A **selfadjoint Hamilton operator**, e. g.

$$H = \tfrac{1}{2m} \bigl(-\mathrm{i}\nabla - A \bigr)^2 + V$$
$$H = m \, \beta + \bigl(-\mathrm{i}\nabla - A \bigr) \cdot \alpha + V$$

2. A **Hilbert space** $\mathcal{H}$ and states are represented by its elements, e. g. $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with $\langle \phi , \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \phi(x) \cdot \psi(x)$.

3. **Dynamics** given by the Schrödinger equation

$$\mathrm{i} \, \partial_t \psi(t) = H \psi(t) , \qquad \psi(0) = \phi$$

Recap: States and Dynamics in Quantum Mechanics

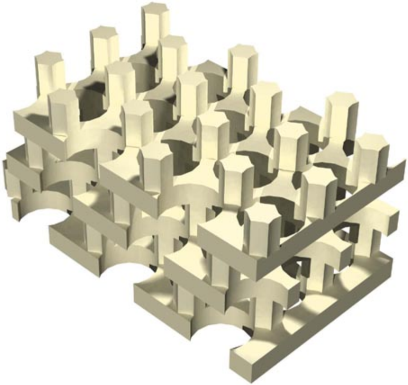
States and Dynamics

1. A **selfadjoint Hamilton operator**
2. A **Hilbert space** $\mathcal{H}$ and states are represented by its elements.
3. The Schrödinger equation

Properties

- $H = H^*$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\phi$
- $\left\|\psi(t)\right\|^2 = \|\phi\|^2$ (conservation of propability)

Maxwell's Equations for Non-Gyrotropic Dielectrics



Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}^{-1}$$

1. $W = \overline{W}$ real (non-gyrotropic)
2. $W^* = W$ (lossless)
3. $0 < c\,\mathbb{1} \leq W \leq C\,\mathbb{1}$ (excludes metamaterials)

Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}(\mathbf{E},\mathbf{H}) = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

2. *Dynamical equations*

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

3. *No sources*

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}(\mathbf{E},\mathbf{H}) = \mathcal{E}(\mathbf{E}(t),\mathbf{H}(t))$$

2. *Dynamical equations*

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

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Schrödinger Formalism of the Maxwell Equations

1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_W(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

$$\left\|(\mathbf{E}, \mathbf{H})\right\|_W^2 = \int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

2. *Dynamical equation* ⇝ "Schrödinger equation"

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$$J_W = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in L^2_W(\mathbb{R}^3, \mathbb{C}^6) \;\middle|\; \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0 \right\}$$

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1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_W(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

$$\big\|(\mathbf{E}, \mathbf{H})\big\|_W^2 = 2\,\mathcal{E}(\mathbf{E}, \mathbf{H})$$

2. *Dynamical equation* ⇝ "Schrödinger equation"

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3. *No sources*

$$J_W = G^{\perp_W}, \qquad G = \text{gradient fields}$$

The Maxwell Operator

$$M = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}$$
$$= W^{-1}\,\mathrm{Rot}$$
$$= -C\,M\,C$$
$$\mathcal{D}(M) = \big(H^1(\mathbb{R}^3,\mathbb{C}^6) \cap \ker \mathrm{Div}\big) \,\widehat{\oplus}\, \mathrm{ran}\,\mathrm{Grad}$$

$M = M^{*_W}$ selfadjoint on *weighted* Hilbert space $L^2_W(\mathbb{R}^3,\mathbb{C}^6)$

$$\langle \Psi, M\Phi \rangle_W = \langle \Psi, W\,W^{-1}\,\mathrm{Rot}\,\Phi \rangle = \langle \mathrm{Rot}\,\Psi, \Psi \rangle$$
$$= \langle W\,M\Psi, \Phi \rangle = \langle M\Psi, W\,\Phi \rangle = \langle M\Psi, \Phi \rangle_W$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}tM}$ **unitary,** leads to **conservation of energy**

The Maxwell Operator

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$$= W^{-1} \, \mathrm{Rot}$$

$$= -C \, M \, C$$

$$\mathcal{D}(M) = \left(H^1(\mathbb{R}^3, \mathbb{C}^6) \cap \ker \mathrm{Div} \right) \widehat{\oplus} \operatorname{ran} \mathrm{Grad}$$

$M = M^{*_W}$ selfadjoint on *weighted* Hilbert space $L^2_W(\mathbb{R}^3, \mathbb{C}^6)$

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$$= \langle W \, M \Psi, \Phi \rangle = \langle M \Psi, W \, \Phi \rangle = \langle M \Psi, \Phi \rangle_W$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}t M}$ **unitary**, leads to **conservation of energy**

The Maxwell Operator

$$
\begin{aligned}
M &= \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} \\
&= W^{-1} \, \mathrm{Rot} \\
&= -C \, M \, C \\
\mathcal{D}(M) &= \mathcal{D}(\mathrm{Rot})
\end{aligned}
$$

$M = M^{*_W}$ selfadjoint on *weighted* Hilbert space $L^2_W(\mathbb{R}^3, \mathbb{C}^6)$

$$
\begin{aligned}
\langle \Psi, M\Phi \rangle_W &= \langle \Psi, W \, W^{-1} \, \mathrm{Rot} \, \Phi \rangle = \langle \mathrm{Rot} \, \Psi, \Psi \rangle \\
&= \langle W \, M\Psi, \Phi \rangle = \langle M\Psi, W \, \Phi \rangle = \langle M\Psi, \Phi \rangle_W
\end{aligned}
$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}t M}$ **unitary**, leads to **conservation of energy**

The Maxwell Operator

$$
\begin{aligned}
M &= \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} \\
&= W^{-1}\,\mathrm{Rot} \\
&= -C\,M\,C \\
\mathcal{D}(M) &= \mathcal{D}(\mathrm{Rot})
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$M = M^{*_W}$ **selfadjoint** on *weighted* Hilbert space $L^2_W(\mathbb{R}^3, \mathbb{C}^6)$

$$
\begin{aligned}
\bigl\langle \Psi, M\Phi \bigr\rangle_W &= \bigl\langle \Psi, W\,W^{-1}\,\mathrm{Rot}\,\Phi \bigr\rangle = \bigl\langle \mathrm{Rot}\,\Psi, \Psi \bigr\rangle \\
&= \bigl\langle W\,M\Psi, \Phi \bigr\rangle = \bigl\langle M\Psi, W\,\Phi \bigr\rangle = \bigl\langle M\Psi, \Phi \bigr\rangle_W
\end{aligned}
$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}t\,M}$ **unitary**, leads to **conservation of energy**

Schrödinger Formalism for Classical Waves

States and Dynamics

1. **“Hamilton” operator** $M = W^{-1}$ Rot where
   - $W = W^*$, $0 < c\,\mathbb{1} \leq W \leq C\,\mathbb{1}$
     (positive, bounded, bounded inverse)
   - Rot $=$ Rot*
2. **Complex (!)** weighted **Hilbert space** $\mathcal{H} \subseteq L^2_W(\mathbb{R}^3, \mathbb{C}^6)$ where

$$\langle \Phi, \Psi \rangle_W = \langle \Phi, W\Psi \rangle = \int_{\mathbb{R}^3} \mathrm{d}x \, \Phi(x) \cdot W(x)\Psi(x)$$

3. **Dynamics** given by **Schrödinger equation**

$$\mathrm{i}\,\partial_t \Psi(t) = M\Psi(t), \qquad \Psi(0) = (\mathbf{E}_0, \mathbf{H}_0)$$

4. **Real-valuedness** of physical solutions

Schrödinger Formalism for Classical Waves

States and Dynamics

1. **"Hamilton" operator** $M = W^{-1}$ Rot with **product structure**
2. **Complex (!)** weighted **Hilbert space** $\mathcal{H} \subseteq L^2_W(\mathbb{R}^3, \mathbb{C}^6)$
3. **Dynamics** given by **Schrödinger equation**
4. **Real-valuedness** of physical solutions

Properties

- $M^{*_W} = M = -C\,M\,C$
- $\mathrm{Re}\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}$ where $\mathrm{Re} = \frac{1}{2}(\mathbb{1} + C)$
  (existence of real solutions)
- $\Psi(t) = \mathrm{e}^{-\mathrm{i}t\,M}\Phi$
- $\left\lVert\Psi(t)\right\rVert^2_W = \lVert\Phi\rVert^2_W$ (conserved quantity, here field energy)

Quantum-Light Analogies

	Wave Equation	**Quantum Mechanics**
Generator dynamics	Maxwell-type operator $M = W^{-1}\,\mathrm{Rot} = M^{*w}$	Hamiltonian $H = -\Delta + V = H^*$
Hilbert space	weighted L^2	L^2
Necessary symmetry	+PH	none
Wave function	$\mathbb{R}$-valued	$\mathbb{C}$-valued
Conserved quantity $\lVert\Psi\rVert^2$	field energy	probability

Doubling of Degrees of Freedom

One of the **tenets of electromagnetism:**

E and **H** are **real** vector fields.

$\Longrightarrow$ Replacing $L^2_W(\mathbb{R}^3, \mathbb{R}^6) \rightsquigarrow L^2_W(\mathbb{R}^3, \mathbb{C}^6)$
doubles the degrees of freedom!

On the other hand, if we want to apply the theory of selfadjoint operators we need to work with **complex** Hilbert spaces!

Restriction to Complex Fields with $\omega > 0$

$C \, M \, C = -M \implies C \, \mathrm{e}^{-\mathrm{i}t \, M} = \mathrm{e}^{-\mathrm{i}t \, M} \, C$ implies

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathrm{e}^{-\mathrm{i}t \, M} \, (\mathbf{E}_0, \mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t \, M} \, (\mathrm{Re} \, \Psi_{\pm}) = \mathrm{Re} \, \left(\mathrm{e}^{-\mathrm{i}t \, M} \Psi_{\pm}\right)$$

where Re $= \frac{1}{2}(\mathbb{1} + C)$ is the real part operator and

$$\begin{aligned} \Psi_{+} &= 1_{\{\omega>0\}}(M) \, (\mathbf{E}_0, \mathbf{H}_0) = P_{+}(\mathbf{E}_0, \mathbf{H}_0) \\ \Psi_{-} &= 1_{\{\omega<0\}}(M) \, (\mathbf{E}_0, \mathbf{H}_0) = P_{-}(\mathbf{E}_0, \mathbf{H}_0) = C \Psi_{+} \end{aligned}$$

are the positive and negative frequency contributions.

Restriction to Complex Fields with $\omega > 0$

$$C\,M\,C = -M \quad\Longrightarrow\quad C\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,C \text{ implies}$$

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0, \mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,(\mathrm{Re}\,\Psi_{\pm}) = \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}t\,M}\Psi_{\pm}\right)$$

$$2\mathrm{Re} = P_+^{-1} \quad\Longrightarrow\quad \text{Study } M_+ := M|_{\mathrm{ran}\,P_+}$$

Real transversal states

$$(\mathbf{E}, \mathbf{H}) = 2\mathrm{Re}\,\Psi_+$$

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

$\longleftrightarrow$

Complex states with $\omega > 0$

$$\Psi_+ = P_+(\mathbf{E}, \mathbf{H})$$

$$\mathrm{i}\,\partial_t \Psi_+ = M_+ \Psi_+$$

Fundamental Constituents

Reduced Description

1. **“Hamilton” operator** $M_+ = W \, \mathrm{Rot}\big|_{\mathrm{ran}\, P_+}$ for $\omega > 0$
2. **Hilbert space** $\mathcal{H}_+ = \mathrm{ran}\, P_+ \subset L^2_W(\mathbb{R}^3, \mathbb{C}^6)$
3. **Dynamics** given by **Schrödinger equation**

$$\mathrm{i}\, \partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H}) \in \mathrm{ran}\, P_+$$

4. **Real-valuedness** of physical solutions:

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 2\mathrm{Re}\, \Psi_+(t)$$

Note

This also applies to **gyrotropic** materials where $W = \left(\begin{smallmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{smallmatrix}\right) \neq \overline{W}$.

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Observables in QM and EM

Quantum Mechanics

- Selfadjoint operators on $\mathcal{H}$
- Wave $\psi(t, x)$ **not** an observable
- *Typical examples:* position $Q = \hat{x}$, momentum $P = -\mathrm{i}\varepsilon\nabla$

Electromagnetism

- Functionals of the fields
- $E_j(t, x)$ and $H_j(t, x)$ **are** observable!

Quadratic Electromagnetic Observables

Examples

- Field components $E_j(t,x)$ and $H_j(t,x)$
- Energy density $e_x(\mathbf{E},\mathbf{H}) = \Psi(x) \cdot W(x)\Psi(x)$
- Poynting vector $\mathcal{P}_x(\mathbf{E},\mathbf{H}) = \frac{1}{2}\mathrm{Re}\left(\overline{\psi^E(x)} \times \psi^H(x)\right)$

$$\Psi = (\psi^E, \psi^H) = P_+(\mathbf{E},\mathbf{H})$$

Quadratic Electromagnetic Observables in Vacuum

$$\big(\mathbf{E}(t,x),\mathbf{H}(t,x)\big) = \mathrm{Re}\left(\Psi(x)\,\mathrm{e}^{-\mathrm{i}t\omega}\right)$$

Examples

- Energy density $e_x(\mathbf{E},\mathbf{H}) = |\Psi(x)|^2$
- Poynting vector
  $\mathcal{P}_x(\mathbf{E},\mathbf{H}) = \mathcal{P}_{O,x}(\mathbf{E},\mathbf{H}) + \mathcal{P}_{S,x}(\mathbf{E},\mathbf{H}) = \tfrac{1}{2}\mathrm{Re}\left(\overline{\psi^E(x)} \times \psi^H(x)\right)$
- In-vacuo spin angular momentum
  $\mathcal{S}_x(\mathbf{E},\mathbf{H}) = \frac{1}{4\omega}\mathrm{Im}\left(\overline{\psi^E(x)} \times \psi^E(x) + \overline{\psi^H(x)} \times \psi^H(x)\right)$
- In-vacuo orbital angular momentum
  $L_x(\mathbf{E},\mathbf{H}) = \frac{1}{4\omega}\mathrm{Im}\left(\psi^E(x)\cdot(x\times\nabla)\psi^E(x) + \overline{\psi^H(x)\cdot(x\times\nabla)\psi^H(x)}\right)$
- In-vacuo orbital momentum
  $\mathcal{P}_{O,x}(\mathbf{E},\mathbf{H}) = \frac{1}{4\omega}\mathrm{Im}\sum_{j=1}^{3}\left(\overline{\psi_j^E(x)}\,\nabla\psi_j^E(x) + \overline{\psi_j^H(x)}\,\nabla\psi_j^H(x)\right)$
- In-vacuo spin momentum $\mathcal{P}_{S,x}(\mathbf{E},\mathbf{H}) = \frac{1}{2}\nabla\times\mathcal{S}_x(\mathbf{E},\mathbf{H})$

Quadratic Electromagnetic Observables in Vacuum

$$\big(\mathbf{E}(t,x),\mathbf{H}(t,x)\big) = \mathrm{Re}\,\Big(\Psi(x)\,\mathrm{e}^{-\mathrm{i}t\omega}\Big)$$

Examples

- Energy density $e_x(\mathbf{E},\mathbf{H}) = |\Psi(x)|^2$
- Poynting vector
  $\mathcal{P}_x(\mathbf{E},\mathbf{H}) = \mathcal{P}_{O,x}(\mathbf{E},\mathbf{H}) + \mathcal{P}_{S,x}(\mathbf{E},\mathbf{H}) = \frac{1}{2}\mathrm{Re}\,\Big(\overline{\psi^E(x)} \times \psi^H(x)\Big)$
- In-vacuo spin angular momentum $\mathcal{S}_{j,x}(\mathbf{E},\mathbf{H}) = \Psi(x)\cdot(\mathbb{1}\otimes S_j)\Psi(x)$
  where $(S_j)_{kn} = -\mathrm{i}\,\epsilon_{jkn}$
- In-vacuo orbital angular momentum $L_x(\mathbf{E},\mathbf{H}) = \Psi(x)\cdot\big(\mathrm{i}(-\mathrm{i}\nabla)\times S\big)\Psi(x)$
- In-vacuo orbital momentum $\mathcal{P}_{O,x}(\mathbf{E},\mathbf{H}) = \Psi(x)\cdot(-\mathrm{i}\nabla)\Psi(x)$
- In-vacuo spin momentum $\mathcal{P}_{S,x}(\mathbf{E},\mathbf{H}) = \nabla\times\Big(\Psi(x)\cdot(\mathbb{1}\otimes S)\Psi(x)\Big)$

Quadratic Electromagnetic Observables in Vacuum

“Quantum expectation values”

Local averages of quadratic observables can be expressed as “quantum expectation values”,

$$\mathcal{F}(\mathbf{E},\mathbf{H}) = |\Lambda|^{-1} \Big\langle P_+(\mathbf{E},\mathbf{H}), 1_\Lambda(x)\, F P_+(\mathbf{E},\mathbf{H}) \Big\rangle_W .$$

These observables are amenable to generalizing quantum mechanical techniques.

- Field energy: $F = \mathbb{1}$
- Spin: $F = S_j$, $(S_j)_{kn} = -\mathrm{i}\,\epsilon_{jkn}$
- Orbital momentum: $F = -\mathrm{i}\nabla = \hat{p}$
- Spin momentum: $F = \mathrm{i}\,\hat{p} \times S$
- Orbital angular momentum: $F = x \times \hat{p}$

Typical States

- **Condensed matter physics:** state is a finite temperature perturbation of the **Fermi projection**

$$P_{\mathrm{F}} = 1_{(-\infty, E_{\mathrm{F}}]}(H)$$

vs.

- **Electromagnetic waves:** Fermi projection unphysical
- **Wave packet states** generated by laser light
- **Spectrally concentrated, multidirectional states** generated by antenna
- Sources play a role (e. g. antenna)

The Bloch-Floquet Transform

Exploiting lattice periodicity: the Bloch-Floquet transform

$$(\mathcal{F}\Psi)(k,x) = \sum_{\gamma\in\Gamma} \mathrm{e}^{-\mathrm{i}k\cdot\gamma}\,\Psi(x+\gamma)$$

where $\Gamma \cong \mathbb{Z}^3$ is the lattice and $k \in \mathcal{B}$ Bloch momentum

The Bloch-Floquet Transform

Exploiting lattice periodicity: the Bloch-Floquet-Zak transform

$$(\mathcal{F}\Psi)(k,x) = \sum_{\gamma\in\Gamma} \mathrm{e}^{-\mathrm{i}k\cdot(x+\gamma)}\,\Psi(x+\gamma)$$

where $\Gamma \cong \mathbb{Z}^3$ is the lattice and $k \in \mathcal{B}$ Bloch momentum

The Frequency Band Picture

$$M \cong \mathcal{F}\, M\, \mathcal{F}^{-1} = \int_{\mathcal{B}}^{\oplus} \mathrm{d}k\, M(k)$$

$$= \int_{\mathcal{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_y + k)^{\times} \\ -(-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix}$$

$$\mathcal{D}\big(M(k)\big) = \underbrace{\big(H^1(\mathbb{T}^3, \mathbb{C}^6) \cap J_{\Gamma}(k)\big)}_{\text{physical states}} \oplus G(k) \subset L^2_W(\mathbb{T}^3, \mathbb{C}^6)$$

$M(k)|_{G(k)} = 0 \Rightarrow$ focus on $M(k)|_{J_{\Gamma}(k)}$

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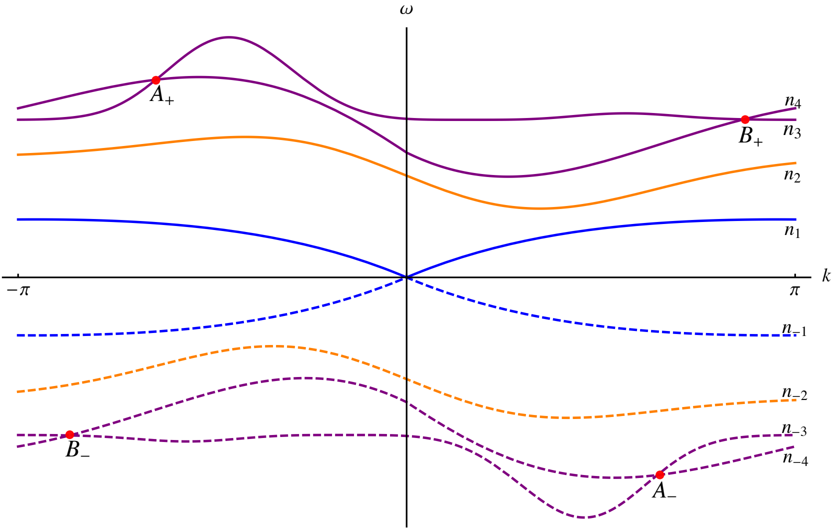
The Frequency Band Picture

Physical bands

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
- Bloch functions $k \mapsto \varphi_n(k)$
- both **locally continuous** everywhere
- both **locally analytic** *away from band crossings*

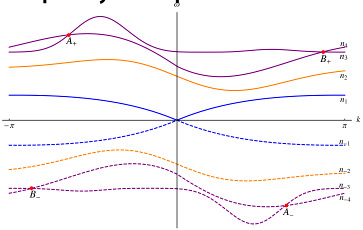
The Frequency Band Picture



Quantum-Light Analogies

photonic crystals $\longleftrightarrow$ **crystalline solids**

Frequency band picture



$\leadsto$ “photonic semiconductor”

Ray optics equations

Onoda et al (2004)
Raghu & Haldane (2006)
De Nittis & L. (2015)

$$\dot{r} = +\nabla_k \Omega + \lambda \, \Xi_{\text{Berry}} \times \dot{k}$$
$$\dot{k} = -\nabla_r \Omega$$

$\Omega(r, k) =$ modified dispersion

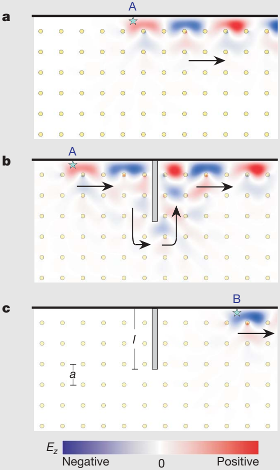
De Nittis & L. (2015): via *“semiclassical” Egorov theorem*

A Novel Class of Materials: *Topological Photonic Crystals*

Predicted theoretically by Raghu & Haldane (2005) ...

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \right\} \Longrightarrow$$

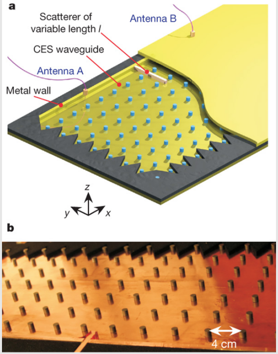
symmetry breaking



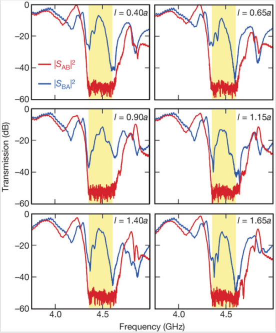
Joannopoulos, Soljačić et al (2009)

A Novel Class of Materials: *Topological Photonic Crystals*

... and realized experimentally by Joannopoulos et al (2009)



Joannopoulos, Soljačić et al (2009)



Joannopoulos, Soljačić et al (2009)

The Quantum Hall Effect: the Prototypical System

$B \neq 0 \implies$ **time-reversal symmetry broken**

$$\sigma^{xy}_{\text{bulk}}(t) \approx \frac{e^2}{h}\,\text{Ch}_{\text{bulk}} = \frac{e^2}{h}\,\text{Ch}_{\text{edge}} \approx \sigma^{xy}_{\text{edge}}(t)$$

transverse conductivity = Chern #

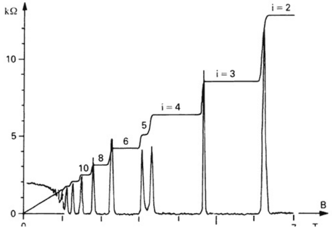
$$\text{Ch}_{\text{bulk/edge}} = \frac{1}{2\pi}\int_{\mathcal{B}} \mathrm{d}k\,\Omega_{\text{bulk/edge}}(k) \in \mathbb{Z}$$

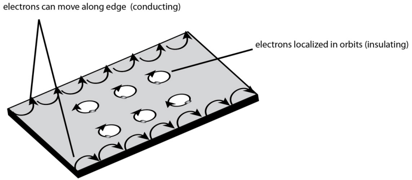
- Edge modes in spectral gaps
- Signed # edge channels $= \text{Ch}(P_{\text{Fermi}})$
- Edge modes unidirectional
- Robust against disorder

***Two* Nobel Prizes**

1985 for experiment: von Klitzing

2016 for theory: Thouless



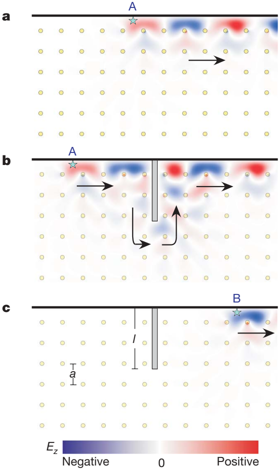


von Klitzing et al (1980)

The Quantum Hall Effect for Light

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \right\} \Longrightarrow$$

symmetry breaking



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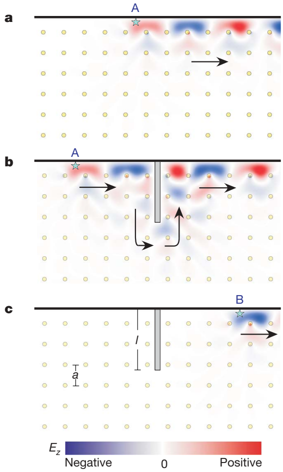

Joannopoulos, Soljačić et al (2009)

- Photonic bulk-edge correspondences

$\downarrow$

- Identify topological observables
  $O = T + \text{error}$

$\downarrow$

- Find all topological invariants T

$\downarrow$

- Classification of PhCs by symmetries

The Quantum Hall Effect for Light

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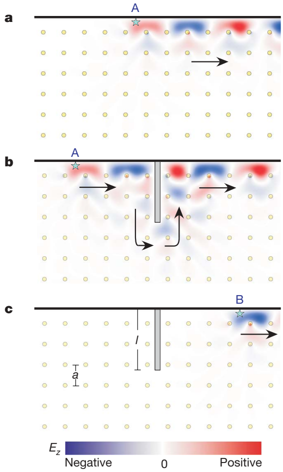

Joannopoulos, Soljačić et al (2009)

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↓

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↓

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↓

- **Classification of PhCs by symmetries**

Question

How many different types of electromagnetic media are there?

Interjection

A Primer on Topological Insulators

Fundamental Notions

Altland–Zirnbauer Classification of Topological Insulators

The 10-fold way

1. **Topological class** of H $\longleftrightarrow$ Symmetries of H
2. $\left.\begin{array}{c}\textbf{Phases}\text{ inside each}\\ \text{topological class}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Labeled by}\\ \text{topological invariants}\end{array}\right.$
3. **Bulk-edge correspondences**

Topological Classes

Symmetries of H $\longleftrightarrow$ **Topological Class** of H

- **Relies on** $\mathrm{i}\partial_t \psi = H\psi$ (Schrödinger equation)
- 3 types of (pseudo) symmetries:
  U unitary/antiunitary, $U^2 = \pm\mathbb{1}$,

$U\,H(k)\,U^{-1} = +H(-k)$ time-reversal symmetry ($\pm$TR)

$U\,H(k)\,U^{-1} = -H(-k)$ particle-hole (pseudo) symmetry ($\pm$PH)

$U\,H(k)\,U^{-1} = -H(+k)$ chiral (pseudo) symmetry (χ)

- $1 + 5 + 4 = 10$ topological classes
- Physics *crucially* depends on topological class.

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- Relies on $\mathrm{i}\partial_t \psi = H\psi$ (Schrödinger equation)
- **3 types of (pseudo) symmetries:**
  U unitary/antiunitary, $U^2 = \pm\mathbb{1}$,

$U\,H(k)\,U^{-1} = +H(-k)$	time-reversal symmetry ($\pm$TR)
$U\,H(k)\,U^{-1} = -H(-k)$	particle-hole (pseudo) symmetry ($\pm$PH)
$U\,H(k)\,U^{-1} = -H(+k)$	chiral (pseudo) symmetry (χ)

- 1 + 5 + 4 = 10 topological classes
- Physics *crucially* depends on topological class.

Phases Inside Topological Classes

- **Inequivalent** phases inside each topological class
- *Continuous*, *symmetry-preserving* deformations of H cannot change topological phase, unless either
  – the energy gap closes (periodic case) or
  – a localization-delocalization transition happens (random case)
- Phases labeled by finite set of **topological invariants** (e. g. Chern numbers but also others)
- **Number and type** of topological invariants determined by
  – *symmetries* $\Longleftrightarrow$ topological class and
  – *dimension* of the system
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Bulk-Edge Correspondences

- Properties on the boundary can be inferred from the bulk
- Consists of 3 equalities:

$$O_{\text{bulk}}(t) \approx T_{\text{bulk}}$$
$$O_{\text{edge}}(t) \approx T_{\text{edge}}$$
$$T_{\text{bulk}} = T_{\text{edge}}$$

- Number and form depends on the topological class
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Bulk-Edge Correspondences

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- Consists of 3 equalities:

$$
\begin{aligned}
O_{\text{bulk}}(t) &\approx T_{\text{bulk}} \\
O_{\text{edge}}(t) &\approx T_{\text{edge}} \\
T_{\text{bulk}} &= T_{\text{edge}}
\end{aligned}
$$

- Number and form depends on the topological class
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Back to Business

Classification of Topological PhCs

Symmetries of Maxwell Operator in Matter

Product structure of $M = W^{-1}\,\mathrm{Rot}$:

$$\left.\begin{aligned} U\,\mathrm{Rot}\,U^{-1} &= \pm\mathrm{Rot} \\ U\,W\,U^{-1} &= \pm W \end{aligned}\right\} \Longrightarrow \quad U\,M\,U^{-1} = \pm M$$

(Signs may be different)

What form do the symmetries U take?

Symmetries of Maxwell Operator in Matter

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(Signs may be different)

What form do the symmetries U take?

Symmetries of the Free Maxwell Operator Rot

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} = -\sigma_2 \otimes \nabla^{\times}$$

Symmetries

For $n = 1, 2, 3$

1. Complex conjugation C (antilinear)
2. $J_n = \sigma_n \otimes \mathbb{1}$ (linear)
3. $T_n = J_n C$ (antilinear)

Admissible Symmetries

$U = C, J_n, T_n$ needs to be a $\langle \cdot , \cdot \rangle_W$-unitary

$$\langle U\Psi, U\Phi \rangle_W = \langle U\Psi, W\, U\Phi \rangle \overset{!}{=} \langle \Psi, W\, \Phi \rangle = \langle \Psi, \Phi \rangle_W$$

$$\Longleftrightarrow$$

$$U^* = U^{-1} \overset{!}{=} U^{*_W} = W^{-1}\, U^*\, W$$

Admissible Symmetries

$U = C, J_n, T_n$ needs to be a $\langle \cdot , \cdot \rangle_W$-unitary

$$\langle U\Psi, U\Phi \rangle_W \overset{!}{=} \langle \Psi, \Phi \rangle_W$$

$$\Longleftrightarrow$$

$$[U, W] = U\,W - W\,U = 0$$

Admissible Symmetries

U maps $\omega > 0$ states onto $\omega > 0$ states

$$U \, P_+ \, U^{-1} = U \, 1_{(0,\infty)}(M) \, U^{-1} = 1_{(0,\infty)}(U \, M \, U^{-1}) \overset{!}{=} P_+ = 1_{(0,\infty)}(M)$$

$$\Longleftrightarrow$$

$$U \, M \, U^{-1} \overset{!}{=} +M$$

Admissible Symmetries

U maps $\omega > 0$ states onto $\omega > 0$ states

$$U \, M \, U^{-1} \overset{!}{=} +M$$

$$\Longleftrightarrow$$

$$U \, \mathrm{Rot} \, U^{-1} \overset{!}{=} +\mathrm{Rot}$$

Admissible Symmetries

Action of symmetries on Rot

1. $C \, \mathrm{Rot} \, C = -\mathrm{Rot}$
2. $J_n \, \mathrm{Rot} \, J_n^{-1} = -\mathrm{Rot}, n = 1, 3$
   $J_2 \, \mathrm{Rot} \, J_2^{-1} = +\mathrm{Rot}$
3. $T_n \, \mathrm{Rot} \, T_n^{-1} = +\mathrm{Rot}, n = 1, 3$
   $T_2 \, \mathrm{Rot} \, T_2^{-1} = -\mathrm{Rot}$

Admissible Symmetries

Action of symmetries on Rot

1. ~~$C\,\mathrm{Rot}\,C = -\mathrm{Rot}$~~
2. ~~$J_n\,\mathrm{Rot}\,J_n^{-1} = -\mathrm{Rot}, n = 1, 3$~~
   $J_2\,\mathrm{Rot}\,J_2^{-1} = +\mathrm{Rot}$
3. $T_n\,\mathrm{Rot}\,T_n^{-1} = +\mathrm{Rot}, n = 1, 3$
   ~~$T_2\,\mathrm{Rot}\,T_2^{-1} = -\mathrm{Rot}$~~

Admissible Symmetries

Lemma (Admissible symmetries)

Only unitary or antiunitary symmetries **commuting** *with* W *and* Rot *are admissible.*

Admissible Symmetries

Lemma (Admissible symmetries)

Writing $W = \sum_{j=0}^{3} \sigma_j \otimes w_j$, *of the aforementioned symmetries* $U = C, J_n, T_n$, $n = 1, 2, 3$, *only three are admissible:*

$U =$	$w_0 =$	$w_1 =$	$w_2 =$	$w_3 =$	*Symmetry Type*
$T_1 = (\sigma_1 \otimes \mathbb{1})\, C$	Re w_0	Re w_1	Re w_2	i Im w_3	*+TR*
$J_2 = \sigma_2 \otimes \mathbb{1}$	w_0	0	w_2	0	*ordinary*
$T_3 = (\sigma_3 \otimes \mathbb{1})\, C$	Re w_0	i Im w_1	Re w_2	Re w_3	*+TR*

Four Types of Electromagnetic Media

Theorem (De Nittis-L. (2016))

There are **4 topologically distinct types** *of media:*

Material	*Realized*	*Conditions on W*	*Symmetries*	*CAZ Class*
Dual symmetric materials & vacuum	*Yes*	$\varepsilon = \mathrm{Re}\,\varepsilon = \mu$, $\chi = -\chi^*$	T_1, T_3, J_2	*N/A*
Non-gyrotropic materials	*Yes*	$\varepsilon = \mathrm{Re}\,\varepsilon$, $\mu = \mathrm{Re}\,\mu$, $\chi = 0$	T_3	*AI*
Gyrotropic materials with dual symmetry	*?*	$\overline{\varepsilon} = \mu$, $\chi = \mathrm{Re}\,\chi$	T_1	*AI*
Gyrotropic materials	*Yes*	*(*$\mathrm{Im}\,\varepsilon \neq 0$ *or* $\mathrm{Im}\,\mu \neq 0$*) and (*$\overline{\varepsilon} \neq \mu$ *or* $\overline{\chi} \neq \chi$*)*	*None*	*A*

Their Topological Classification

Corollary (Class AI media (non-gyrotropic and gyrotropic, dual-symmetric media))

In $d = 2, 3$ media are topologically trivial, i. e. there is a single phase.

In $d = 4$ phases are labeled by the $\mathbb{Z}$-valued second Chern number.

Their Topological Classification

Corollary (Class A media (gyrotropic media))

Phases are labeled by $\mathbb{Z}$-valued Chern numbers, in

$d = 2$ *by a single first Chern number.*

$d = 3$ *by three first Chern numbers.*

$d = 4$ *by six first Chern numbers and one second Chern number.*

Their Topological Classification

Theorem (Dual symmetric media, Gomi, De Nittis & L. (2017))

In $d = 2, 3$ these media are trivial, i. e. there is a single phase.

The case $d = 4$ is work in progress.

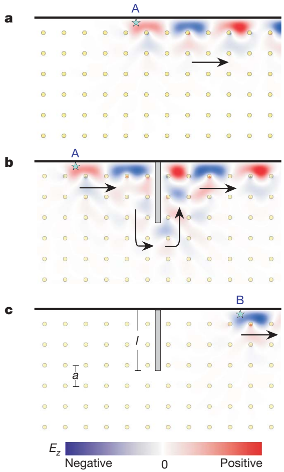
The Quantum Hall Effect for Light

Progress

- Classification of medium: CAZ class A
- Topological invariant: Chern number

Work in progress

- Topological bulk/edge observables?
  What is the photonic analog of the transverse conductivity?
- Bulk-edge correspondence
- **Quantitative** physical understanding



Joannopoulos, Soljačić et al (2009)

Finding Physically Measurable, Conserved Quantities

Example to Illustrate the Problem

Local Energy Conservation Law

$$\partial_t \mathcal{E}_x\big(\mathbf{E}(t), \mathbf{H}(t)\big) + \nabla \cdot \mathcal{P}_x\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 0$$

with energy density

$$\mathcal{E}_x(\mathbf{E}, \mathbf{H}) = \Psi(x) \cdot W(x)\Psi(x)$$

and Poynting vector

$$(\mathbf{E}, \mathbf{H}) = \frac{1}{2}\mathrm{Re}\left(\overline{\psi^E(x)} \times \psi^H(x)\right)$$

where $\Psi(x) = P_+(\mathbf{E}, \mathbf{H})$

"Gauge" freedom: $\nabla \cdot (\nabla \times \mathbf{E}) = 0$!

Finding Physically Measurable, Conserved Quantities

Example to Illustrate the Problem

Local Energy Conservation Law

$$\partial_t \mathcal{E}_x\big(\mathbf{E}(t), \mathbf{H}(t)\big) + \nabla \cdot \mathcal{P}'_x\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 0$$

with energy density

$$\mathcal{E}_x(\mathbf{E}, \mathbf{H}) = \Psi(x) \cdot W(x)\Psi(x)$$

and Poynting vector plus a rotational component

$$\mathcal{P}'_x(\mathbf{E}, \mathbf{H}) = \frac{1}{2}\mathrm{Re}\left(\overline{\psi^E(x)} \times \psi^H(x)\right) + \nabla \times \mathcal{F}_x(\mathbf{E}, \mathbf{H})$$

where $\Psi(x) = P_+(\mathbf{E}, \mathbf{H})$

"Gauge" freedom: $\nabla \cdot (\nabla \times \mathbf{E}) = 0$!

Finding Physically Measurable, Conserved Quantities

Example to Illustrate the Problem

Local Energy Conservation Law in Vacuum

$$\partial_t \mathcal{E}_x\big(\mathbf{E}(t), \mathbf{H}(t)\big) + \nabla \cdot \mathcal{P}'_x\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 0$$

with energy density

$$\mathcal{E}_x(\mathbf{E}, \mathbf{H}) = \left|\Psi(x)\right|^2$$

and Poynting vector plus a rotational component

$$\mathcal{P}'_x(\mathbf{E}, \mathbf{H}) = \frac{1}{2} \mathrm{Re}\left(\overline{\psi^E(x)} \times \psi^H(x)\right) + \nabla \times \mathcal{F}_x(\mathbf{E}, \mathbf{H})$$

where $\Psi(x) = P_+(\mathbf{E}, \mathbf{H})$

“Gauge” freedom: $\nabla \cdot (\nabla \times \mathbf{E}) = 0$!

Finding Physically Measurable, Conserved Quantities

We have already seen *two* candidates, the **Poynting vector**

$$\mathcal{P}_x(\mathbf{E},\mathbf{H}) = \frac{1}{2}\mathrm{Re}\left(\overline{\psi^E(x)} \times \psi^H(x)\right) = \mathcal{P}_{O,x}(\mathbf{E},\mathbf{H}) + \mathcal{P}_{S,x}(\mathbf{E},\mathbf{H})$$

and the **orbital contribution** to the field's momentum

$$\mathcal{P}_{O,x,j}(\mathbf{E},\mathbf{H}) = \Psi(x) \cdot \big(-\mathrm{i}\partial_{x_j}\Psi\big)(x).$$

The spin contribution

$$\mathcal{P}_{S,x,j}(\mathbf{E},\mathbf{H}) = \nabla \times \Big(\Psi(x) \cdot (\mathbb{1} \otimes S_j)\Psi(x)\Big), \quad (S_j)_{kn} = -\mathrm{i}\epsilon_{jkn},$$

does not contribute to energy transport.

Do physicists measure $\mathcal{P}_x$ or $\mathcal{P}_{O,x}$?

Finding Physically Measurable, Conserved Quantities

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Finding Physically Measurable, Conserved Quantities

Topic of active research

New Journal of Physics
The open access journal for physics

Dual electromagnetism: helicity, spin, momentum and angular momentum

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Optical chirality in gyrotropic media: symmetry approach

Igor Proskurin,[1,2,*] Alexander S. Ovchinnikov,[2] Pavel Nosov,[2] and Jun-ichiro Kishine[3]

Lagrangian Formalism & Noether's Theorem

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}+\nabla\times\mathbf{H}\\ -\nabla\times\mathbf{E}\end{pmatrix}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{E}\\ \nabla\cdot\mathbf{H}\end{pmatrix} = 0$$

- In vacuo Maxwell equations: *the* relativistic equations for a massless spin-1 particle
- Symmetry group: **Poincaré group**

Lagrangian Formalism & Noether's Theorem

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$$\begin{pmatrix}\nabla \cdot \mathbf{E}\\ \nabla \cdot \mathbf{H}\end{pmatrix} = 0$$

- Focus on **dual symmetry**

$$\mathrm{i}J_2 : (\mathbf{E}, \mathbf{H}) \mapsto (\mathrm{i}\sigma_2 \otimes \mathbb{1})(\mathbf{E}, \mathbf{H}) = (\mathbf{H}, -\mathbf{E})$$

- Continuous version:

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} \mapsto \mathrm{e}^{\mathrm{i}\theta J_2}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

- Dual symmetry $\Longrightarrow$ conservation of helicity

Lagrangian Formalism & Noether's Theorem

Standard Lagrangian density for electromagnetism

$$\mathcal{L}(E, H) = E^2 - H^2$$

and we express

$$\mathbf{E} = -\partial_t \mathbf{A}$$
$$\mathbf{H} = \nabla \times \mathbf{A}$$

in terms of gauge potentials (in Coulomb gauge $\nabla \cdot \mathbf{A} = 0$).

Lagrangian Formalism & Noether's Theorem

Standard Lagrangian density for electromagnetism

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Obtaining conserved quantities: Noether's Theorem

- Rotational symmetry
  $\rightsquigarrow$ conservation of **faux** total angular momentum

$$\begin{aligned}\widetilde{J} &= \widetilde{L} + \widetilde{S} \\ &= \mathbf{E} \cdot (x \times \nabla)\mathbf{A} + \mathbf{E} \times \mathbf{A}\end{aligned}$$

- Faux orbital angular momentum $\widetilde{L}$ $\rightsquigarrow$ *not* conserved!
- Faux helicity $\widetilde{S}$ $\rightsquigarrow$ *not* conserved!
- Do $\widetilde{J}$, $\widetilde{L}$ and $\widetilde{S}$ correspond to measurable quantities?

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$$\text{Conserved} = \text{non-conserved} + \text{non-conserved}$$
$$= \mathbf{E} \cdot (x \times \nabla)\mathbf{A} + \mathbf{E} \times \mathbf{A}$$

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Lagrangian Formalism & Noether's Theorem

Standard Lagrangian density for electromagnetism

$$\mathcal{L}(E, H) = E^2 - H^2$$

Why?

- Lagrangian density breaks dual symmetry
- Physical system still has this symmetry, but no longer obvious in choice of Lagrangian
- Integrating faux and proper total angular momentum yields same quantity

$$\int_{\mathbb{R}^3} \mathrm{d}x \, J = \int_{\mathbb{R}^3} \mathrm{d}x \, \widetilde{J} = \text{const}$$

Dual symmetric approach by Bliokh et al

Idea

Use a dual-symmetric Lagrangian density

- Consider $F = (\mathbf{E}, \mathbf{H})$ and $G = (\mathbf{H}, -\mathbf{E})$ as independent variables
- Corresponds to complexifying the electromagnetic field!
- Necessary: eigenvalues and eigenvectors of $\mathrm{i}\sigma_2$ complex

Dual symmetric approach by Bliokh et al

Idea

Use a dual-symmetric Lagrangian density

- $\mathbf{E} = -\nabla \times \mathbf{C} = -\partial_t \mathbf{A}$ and $\mathbf{H} = \nabla \times \mathbf{A} = -\partial_t \mathbf{C}$ and $\nabla \cdot \mathbf{A} = 0 = \nabla \cdot \mathbf{C}$ (transversal gauge)
- Yields physically meaningful and conserved angular momenta

$$J = L + S$$

$$L = \frac{1}{2}\Big(\mathbf{E} \cdot (x \times \nabla)\mathbf{A} + \mathbf{H} \cdot (x \times \nabla)\mathbf{C}\Big)$$

$$S = \frac{1}{2}\Big(\mathbf{E} \times \mathbf{A} + \mathbf{H} \times \mathbf{C}\Big)$$

Difficulties in the Lagrangian Formalism

- **Arbitrariness** in choice of Lagrangian density
  $\rightsquigarrow$ Which one is "better"?
- Formulation involves **potentials**
- Symmetry $\rightsquigarrow$ conserved quantity
- However: Symmetry $\rightsquigarrow$ **physically meaningful quantity**?!?
- Remember: this is all in **vacuum**! (Simplest case!)
  $\rightsquigarrow$ media?

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Conserved Quantities in the Schrödinger Formalism

Consider quadratic observables of the form

$$\mathcal{F}\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \Big\langle P_+\big(\mathbf{E}(t), \mathbf{H}(t)\big), F\, P_+\big(\mathbf{E}(t), \mathbf{H}(t)\big) \Big\rangle_W$$

where $F = F^{*_W}$.

Simple Criterion

$$\mathcal{F} \text{ conserved} \Longleftrightarrow [F, M_+] = 0$$

↝ *More* general than Noether's Theorem, also includes discrete symmetries!

Conserved Quantities in the Schrödinger Formalism

Consider quadratic observables of the form

$$\mathcal{F}\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \Big\langle \mathrm{e}^{-\mathrm{i}t M_+} \Phi_+ \, , \, F \, \mathrm{e}^{-\mathrm{i}t M_+} \Phi_+ \Big\rangle_W$$

where $F = F^{*_W}$ and $\Phi_+ = P_+(\mathbf{E}_0, \mathbf{H}_0)$.

Simple Criterion

$$\mathcal{F} \text{ conserved} \Longleftrightarrow [F, M_+] = 0$$

⇝ *More* general than Noether's Theorem, also includes discrete symmetries!

Conservation of Helicity in Matter

Lemma (Conservation of helicity aka Lipkin zilch)

Suppose $[W, J_2] = 0$ *holds. Then*

$$\mathcal{H}\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \Big\langle \Psi(t), J_2\, M_+\, \Psi(t) \Big\rangle_W = \Big\langle \Psi(0), J_2\, M_+\, \Psi(0) \Big\rangle_W$$

Proof.

$$\begin{aligned}
\mathcal{H}\big(\mathbf{E}(t), \mathbf{H}(t)\big) &= \Big\langle \Psi(t), J_2\, M_+\, \Psi(t) \Big\rangle_W \\
&= \Big\langle \mathrm{e}^{-\mathrm{i}t\, M_+}\, \Psi(0), J_2\, M_+\, \mathrm{e}^{-\mathrm{i}t\, M_+}\, \Psi(0) \Big\rangle_W \\
&= \Big\langle \mathrm{e}^{-\mathrm{i}t\, M_+}\, \Psi(0), \mathrm{e}^{-\mathrm{i}t\, M_+}\, J_2\, M_+\, \Psi(0) \Big\rangle_W \\
&= \Big\langle \Psi(0), J_2\, M_+\, \Psi(0) \Big\rangle_W = \mathcal{H}\big(\mathbf{E}(0), \mathbf{H}(0)\big)
\end{aligned}$$

□

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Proof.

$$\begin{aligned}
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&= \Big\langle \mathrm{e}^{-\mathrm{i} t M_+} \, \Psi(0), J_2 \, M_+ \, \mathrm{e}^{-\mathrm{i} t M_+} \, \Psi(0) \Big\rangle_W \\
&= \Big\langle \mathrm{e}^{-\mathrm{i} t M_+} \, \Psi(0), \mathrm{e}^{-\mathrm{i} t M_+} \, J_2 \, M_+ \, \Psi(0) \Big\rangle_W \\
&= \Big\langle \Psi(0), J_2 \, M_+ \, \Psi(0) \Big\rangle_W = \mathcal{H}\big(\mathbf{E}(0), \mathbf{H}(0)\big)
\end{aligned}$$

□

Comparison of Both Approaches

Schrödinger

- Inherently dual symmetric
- Does not involve potentials
- Generalization to media straightforward
- Easier to guess physical observables

Lagrange

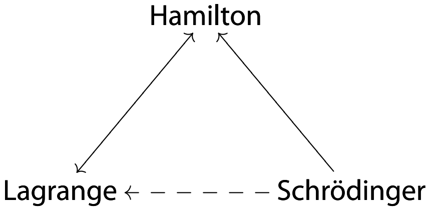
- Not obvious in standard approach
- Involves potentials
- Conserved quantities physical?

Why study the Schrödinger formalism?

Viewing one system in different ways.

Access to tools and techniques.

Three Frameworks to in Which to View Physical Systems



- Hamiltonian and Lagrangian mechanics equivalent (subject to mathematical conditions)
- Schrödinger equation gives rise to linear Hamiltonian system of equations
- Every framework has its own tools

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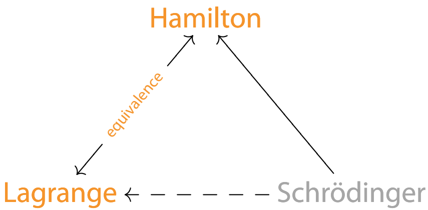

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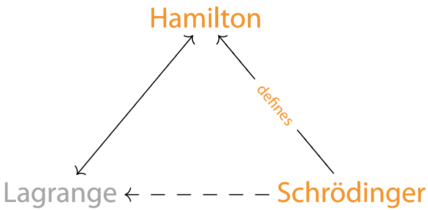

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Schrödinger Equation $\rightarrow$ Hamiltonian Equations

Definition (Symplectic Vector Space)

A **symplectic vector space** $(\mathcal{V}, \Omega)$ is a pair composed of a **real Banach space** $\mathcal{V}$ endowed with a **symplectic form**, i. e. a non-degenerate, antisymmetric bilinear form $\Omega : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$.

Schrödinger Equation $\rightarrow$ Hamiltonian Equations

Definition (Linear Hamiltonian System)

Let $(\mathcal{V}, \Omega)$ be a **symplectic vector space** and $A : \mathcal{V} \longrightarrow \mathcal{V}$ a linear operator that is Ω-skew, $\Omega(v, Aw) = -\Omega(Av, w)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = A v(t), \qquad v(0) = w \in \mathcal{V},$$

is a linear Hamiltonian system of equations.

Schrödinger Equation $\rightarrow$ Hamiltonian Equations

Proposition (Marsden & Ratiu, Corollary 2.5.2)

*Suppose $H = H^*$ is a **selfadjoint (hermitian) operator** on a **complex Hilbert space** $\mathcal{H}$ with **scalar product** $\langle \cdot \, , \, \cdot \rangle$ that enters the **Schrödinger equation***

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi(t) = H \psi(t), \qquad \psi(0) = \phi \in \mathcal{H} .$$

*Then this defines a **Linear Hamiltonian System** where $\mathcal{V} = \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}} \simeq \mathcal{H}$ (identifying $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$), **symplectic form***

$$\Omega(\psi, \phi) = -2\hbar \, \mathrm{Im} \, \langle \psi, \phi \rangle \, ,$$

and the operator $A = \mathrm{i}\hbar H$.

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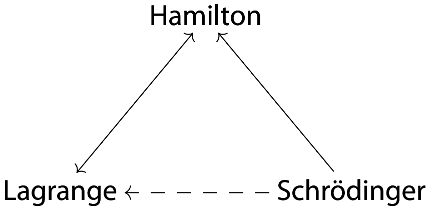

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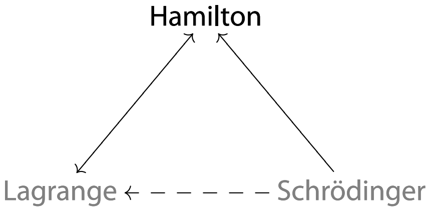
Example: Finding Constants of Motion



Find an observable F so that

- $\{\cdot\,,\,\cdot\}$ Poisson bracket, $\{H, F\} = 0$
- $[H, F] = H\,F - F\,H = 0$
- Noether's theorem $\rightsquigarrow$ yields conserved quantity

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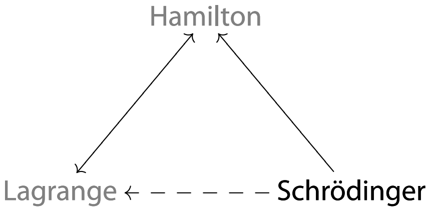

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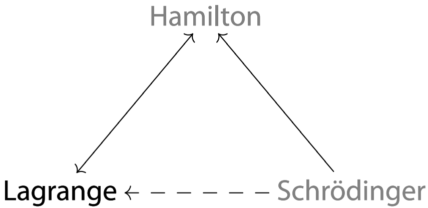

Find a continuous symmetry

- $\{\cdot\,,\,\cdot\}$ Poisson bracket, $\{H, F\} = 0$
- $[H, F] = H\,F - F\,H = 0$
- Noether's theorem $\rightsquigarrow$ yields conserved quantity

Structural Similarities of Certain Classical Wave Equations

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \, \rho_0 \\ -\rho_0^{-1} \nabla \, \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^{\dagger}(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^{\dagger}(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^n$

Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

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Video

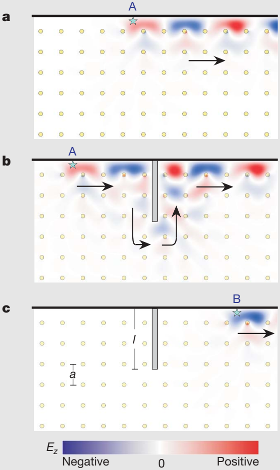
Unidirectional Edge Modes in Coupled Mechanical Oscillators

The Quantum Hall Effect for Light

Predicted theoretically by Raghu & Haldane (2005) ...

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \quad \right\} \quad \Longrightarrow$$

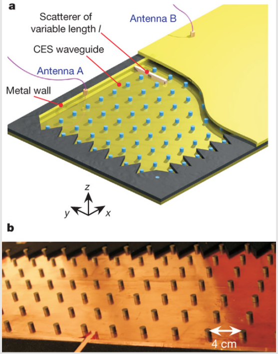
symmetry breaking



Joannopoulos, Soljačić et al (2009)

The Quantum Hall Effect for Light

... and realized experimentally by Joannopoulos et al (2009)



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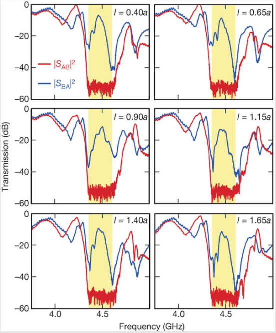


Joannopoulos, Soljačić et al (2009)

The Quantum Hall Effect: the Prototypical System

$$B \neq 0 \implies \text{time-reversal symmetry broken}$$

$$\sigma^{xy}_{\text{bulk}}(t) \approx \frac{e^2}{h}\,\mathrm{Ch}_{\text{bulk}} = \frac{e^2}{h}\,\mathrm{Ch}_{\text{edge}} \approx \sigma^{xy}_{\text{edge}}(t)$$

transverse conductivity = Chern #

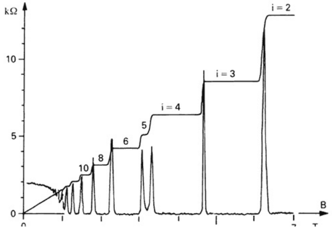
$$\mathrm{Ch}_{\text{bulk/edge}} = \frac{1}{2\pi}\int_{\mathcal{B}} \mathrm{d}k\,\Omega_{\text{bulk/edge}}(k) \in \mathbb{Z}$$

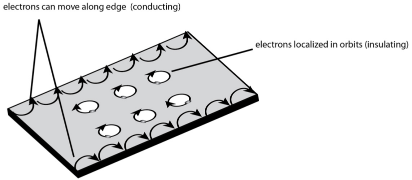
- Edge modes in spectral gaps
- Signed # edge channels $= \mathrm{Ch}(P_{\text{Fermi}})$
- Edge modes unidirectional
- Robust against disorder

***Two* Nobel Prizes**

1985 for experiment: von Klitzing

2016 for theory: Thouless

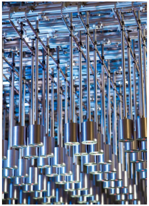




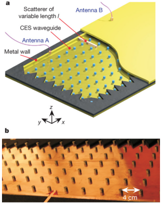
von Klitzing et al (1980)

Claim: Three Experiments are
Different Manifestations of Same Underlying Physical Principles.

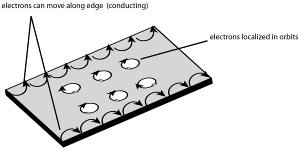
Phenomenological Similarities



Coupled Oscillators



Light



Quantum

- Periodic structure
- Breaking of time-reversal symmetries
- Boundary modes
- Robust under perturbations

Making the Pononic QHE Analogy Rigorous

- Schrödinger formalism coupled mechanical oscillators
- Experiment and topological classification due to Süsstrunk & Huber (ETH Zürich)
- Reality condition/restriction to $\omega > 0$ not taken into account
- $\Longrightarrow$ **Unphysical particle-hole and chiral symmetries**

> Current Issue > vol. 113 no. 33 > Roman Süsstrunk, E4767–E4775

Check for updates

Classification of topological phonons in linear mechanical metamaterials

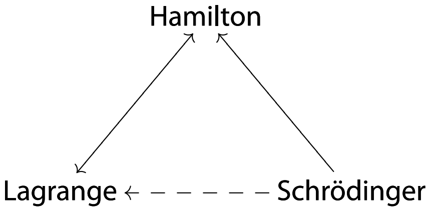
Roman Süsstrunk[a,1] and Sebastian D. Huber[a]

Advances in Understanding of Quantum Systems

- Spectral theory
- Scattering theory
- Semiclassical limits
- Perturbation theory
- Non-linear effects
- Periodic operators
- Random operators
- Topological insulators

Adapt and apply these techniques to classical wave equations

Today's Goals



1. Derive **Schrödinger formalism for classical electromagnetic waves**
2. Application of Schrödinger formalism: Classify **topological photonic crystals**
3. Comparing Schrödinger and Lagrangian formalism: finding **constants of motion** in electromagnetism

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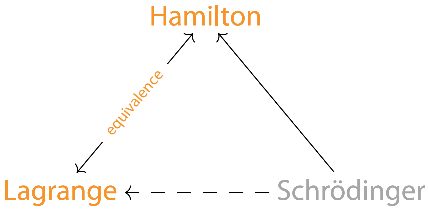

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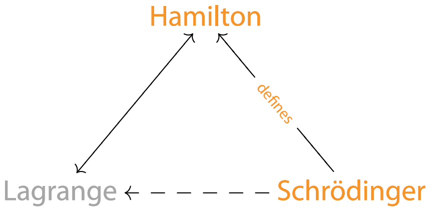

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Thank you very much