“Semiclassical” Ray Optics in Photonic Crystals

in collarboration with Giuseppe De Nittis

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Idea

Realizing Quantum Effects with Classical Waves

Goal of Today's Talk

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

(dynamical equation)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

$$\xrightarrow{\lambda \ll 1} \begin{cases} \dot{r} = +\nabla_k \Omega + \mathcal{O}(\lambda) \\ \dot{k} = -\nabla_r \Omega + \mathcal{O}(\lambda) \end{cases}$$

(ray optics equations)

Setting

- Perturbation parameter $\lambda \ll 1$
- Slowly varying *electric permittivity* $\varepsilon = \varepsilon(\lambda)$ and *magnetic permeability* $\mu = \mu(\lambda)$ are 3×3-matrix-valued
- ε and μ: periodic to “leading order”

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(ray optics equations)

Goal

Given a particular initial state $(\mathbf{E}_0, \mathbf{H}_0)$, find dispersion relation $\Omega(r,k)$ and $\mathcal{O}(\lambda)$ terms.

Similar physics, similar mathematics?

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \, \rho_0 \\ -\rho_0^{-1} \nabla \, \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^N$

Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

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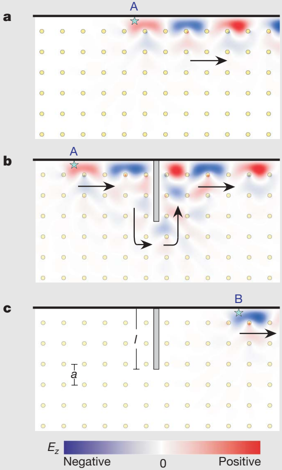
Examples of Quantum-Wave Analogies

- Periodic media $\longleftrightarrow$ crystalline solids (periodic operators, Bloch-Floquet theory)
- Random media $\longleftrightarrow$ random Schrödinger operators
- **"Topological Insulators" for classical waves**
  (due to Haldane, 2016 Nobel Prize in Physics!)

Quantum Hall Effect for Light

Predicted theoretically by Raghu & **Haldane** (2005) ...

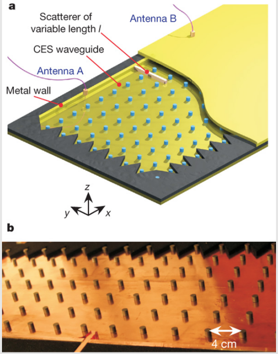
$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \quad \text{symmetry breaking} \right\} \Longrightarrow$$



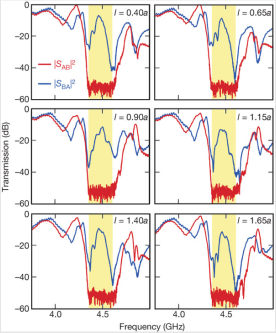
Joannopoulos, Soljačić et al (2009)

Quantum Hall Effect for Light

... and realized experimentally by Joannopoulos et al (2009)



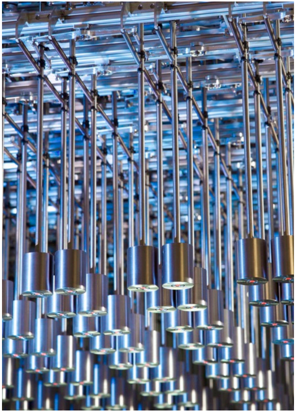
Joannopoulos, Soljačić et al (2009)



Joannopoulos, Soljačić et al (2009)

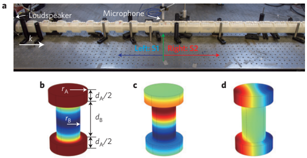
Topological Insulators for Other Waves: Experiments

Mechanical



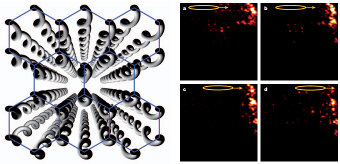
Süsstrunk & Huber (2015)

Acoustic



Xiao, Ma et al (2015)

Periodic Waveguide Arrays



Rechtsman, Szameit et al (2013)

Despite Experiments ...

... **first-principle derivations** are scarce, be it rigorous or non-rigorous!

$\rightsquigarrow$ Open field with lots of interesting problems!

Making Quantum Analogies Rigorous

Develop and explore the **Schrödinger formalism for** certain **classical wave equations**

- Allows for adaptation of techniques from quantum mechanics to other wave equations
- Also differences, e. g. classical waves $\mathbb{R}$-valued

Today: Derivation of ray optics equations via semiclassical techniques

Making Quantum Analogies Rigorous

Develop and explore the **Schrödinger formalism for** certain **classical wave equations**

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Today: **Derivation of ray optics equations via semiclassical techniques**

Recap: Schrödinger Equation on $\mathbb{R}^d$

Fundamental Constituents

1. Hamilton/Schrödinger operator H, typical examples are

$$H = \tfrac{1}{2m}\bigl(-\mathrm{i}\nabla - A\bigr)^2 + V$$
$$H = m\,\beta + \bigl(-\mathrm{i}\nabla - A\bigr) \cdot \alpha + V$$

2. Hilbert space $L^2(\mathbb{R}^d, \mathbb{C}^n)$ where $\langle \phi, \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \phi(x) \cdot \psi(x)$

3. Dynamics given by Schrödinger equation

$$\mathrm{i}\partial_t \psi(t) = H\psi(t), \qquad \psi(0) = \phi$$

Recap: Schrödinger Equation on $\mathbb{R}^d$

Fundamental Constituents

1. Hamilton/Schrödinger operator H
2. Hilbert space
3. Schrödinger equation

Properties

- $H = H^*$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\phi$
- $\left\|\psi(t)\right\|^2 = \|\phi\|^2$ (conservation of propability)

Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* $M = W \, D$ where
   - $W = W^*$, $0 < c \, \mathrm{id} \leq W \leq C \, \mathrm{id}$
     (positive, bounded, bounded inverse)
   - $D = D^*$ (potentially unbounded)

2. Complex (!) *Hilbert space* $\mathfrak{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$ where

$$\langle \phi, \psi \rangle_W = \langle \phi, W^{-1} \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \phi(x) \cdot W^{-1} \psi(x)$$

3. Dynamics given by *Schrödinger equation*

$$\mathrm{i} \partial_t \psi(t) = M \psi(t), \qquad \psi(0) = \phi$$

4. *Even particle-hole symmetry* K, i. e.
   K antiunitary, $K^2 = +\mathrm{id}$ and $K \, M \, K = -M$

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2. Complex (!) **weighted** *Hilbert space* $\mathfrak{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$ where

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Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* $M = W\,D$ with **product structure**
2. Complex (!) **weighted** *Hilbert space* $\mathfrak{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$
3. Dynamics given by *Schrödinger equation*
4. *Even particle-hole symmetry* K

Properties

- $M^{*_W} = M$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}t\,M}\phi$
- $\big\|\psi(t)\big\|^2_W = \|\phi\|^2_W$ (conserved quantity, e. g. energy)
- $\mathrm{Re}_K\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}_K$ where $\mathrm{Re}_K = \frac{1}{2}(\mathrm{id} + K)$
(existence of real solutions)

Quantum-Wave Analogies

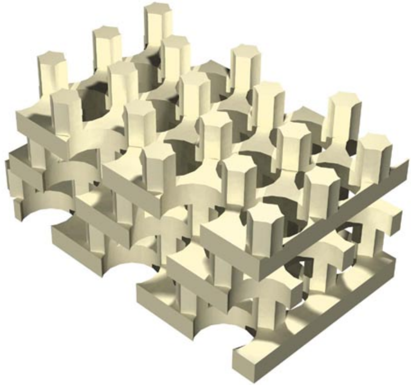
	Wave Equation	**Quantum Mechanics**
Hilbert space	weighted L^2	L^2
Wave function	$\mathbb{R}$-valued	$\mathbb{C}$-valued
Generator dynamics	Maxwell-type operator $M = W\,D = M^*$	Hamiltonian $H = \hat{p}^2 + V = H^*$
Necessary symmetry	+PH	none
Conserved quantity $\lVert\Psi\rVert^2$	e. g. field energy	probability

Aim of this Section

Make a first-principles derivation of the Schrödinger formalism for electromagnetic waves, i. e. identify

1. "Hamilton" operator $M = W\,D$
2. Hilbert space
3. Schrödinger equation
4. Even particle-hole symmetry

Maxwell's Equations for Non-Gyrotropic Dielectrics



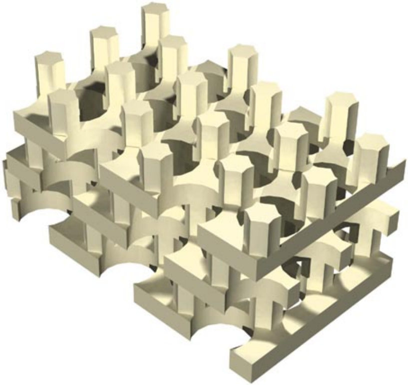
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

1. $W = \overline{W}$ real (non-gyrotropic)
2. $W^* = W$ (lossless)
3. $0 < c\,\mathbf{1} \leq W \leq C\,\mathbf{1}$ (excludes metamaterials)
4. W frequency-independent (no dispersion)

Maxwell's Equations for Non-Gyrotropic Dielectrics



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Schrödinger Formalism of Electromagnetism

$$\left.\begin{matrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} \\ \text{dynamical Maxwell equations}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} \mathrm{i}\partial_t \Psi = M \Psi \\ \text{“Schrödinger-type equation”}\end{matrix}\right.$$

$$\Psi(t) = \big(\mathbf{E}(t), \mathbf{H}(t)\big) \in \mathfrak{H} = \Big\{ \Psi \in L^2_W(\mathbb{R}^3, \mathbb{C}^6) \;\big|\; \Psi \text{ transversal} \Big\}$$

$$M = \underbrace{\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1}}_{= W^{-1}} \underbrace{\begin{pmatrix} 0 & +(-\mathrm{i}\nabla)^{\times} \\ -(-\mathrm{i}\nabla)^{\times} & 0 \end{pmatrix}}_{= \mathrm{Rot}} = M^*$$

$$\left.\begin{matrix} \text{Maxwell equations} \\ \Longleftrightarrow \\ \text{Maxwell operator } M = M^* \end{matrix}\right\} \Longrightarrow$$

Adaptation of **techniques from quantum mechanics** to electromagnetism

Fundamental Symmetries of Non-Gyrotropic Materials

$$\left.\begin{matrix}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}=\begin{pmatrix}-\nabla\times\mathbf{H}\\ +\nabla\times\mathbf{E}\end{pmatrix}\\ \text{dynamical Maxwell equations}\end{matrix}\right\}\iff\left\{\begin{matrix}\mathrm{i}\partial_t\Psi = M\Psi\\ \text{“Schrödinger-type equation”}\end{matrix}\right.$$

$$W(x)=\begin{pmatrix}\varepsilon(x) & 0\\ 0 & \mu(x)\end{pmatrix}=\begin{pmatrix}\overline{\varepsilon(x)} & 0\\ 0 & \overline{\mu(x)}\end{pmatrix}=\overline{W(x)}$$

3 Symmetries

1. $C:(\mathbf{E},\mathbf{H})\mapsto(\overline{\mathbf{E}},\overline{\mathbf{H}})$ with $C\,M\,C=-M$ (+PH)
2. $J:(\mathbf{E},\mathbf{H})\mapsto(\mathbf{E},-\mathbf{H})$ with $J\,M\,J=-M$ (χ)
3. $T=J\,C:(\mathbf{E},\mathbf{H})\mapsto(\overline{\mathbf{E}},-\overline{\mathbf{H}})$ with $T\,M\,T=+M$ (+TR)

Restriction to Real Fields

$C\,M\,C = -M$ implies

$$\mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0,\mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}\,\Psi_{\pm} = \mathrm{Re}\;\mathrm{e}^{-\mathrm{i}t\,M}\Psi_{\pm}$$

where Re $= \frac{1}{2}(\mathrm{id} + C)$ is the real part operator and

$$\Psi_+ = 1_{\{\omega>0\}}(M)\,(\mathbf{E}_0,\mathbf{H}_0) = P_+(\mathbf{E}_0,\mathbf{H}_0)$$
$$\Psi_- = 1_{\{\omega<0\}}(M)\,(\mathbf{E}_0,\mathbf{H}_0) = P_-(\mathbf{E}_0,\mathbf{H}_0) = C\Psi_+$$

the positive and negative frequency contributions

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the positive and negative frequency contributions

What About Gyrotropic Media?

What if

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \neq \begin{pmatrix} \overline{\varepsilon(x)} & 0 \\ 0 & \overline{\mu(x)} \end{pmatrix} = \overline{W(x)}$$

is complex?

1. Use non-gyrotropic equations $\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$
   $\rightsquigarrow$ often implicitly use in literature, but $\mathrm{Im}\,\big(\mathbf{E}(t), \mathbf{H}(t)\big) \neq 0$ ⚡
2. Use $(\mathbf{E}, \mathbf{H}) = \frac{1}{2}(\Psi_+ + \Psi_-)$ and let positive/negative frequency contributions evolve separately via $M = M_+ \oplus M_-$

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The Schrödinger Formalism for Gyrotropic Media

$\Psi_-(t) = C\Psi_+(t)$ can be enforced by choosing $W_- = \overline{W_+}$, i. e.

$$M_\pm = -C\, M_\mp\, C = W_\pm \, \mathrm{Rot} \,\big\vert_{\pm\omega>0}$$

Relation between $M_\pm$ implies relation between evolution groups:

$$C\, \mathrm{e}^{-\mathrm{i} M_\pm} = \mathrm{e}^{-\mathrm{i} t\, M_\mp}\, C$$

The Schrödinger Formalism for Gyrotropic Media

Maxwell equations equivalent to

$$\mathrm{i}\partial_t \Psi(t) = M \Psi(t), \qquad \Psi(0) = \Phi \in \mathfrak{H},$$

on the Hilbert space

$$\mathfrak{H} := \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_- \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) \oplus L^2_{W_-}(\mathbb{R}^3, \mathbb{C}^6)$$

with Maxwell operator

$$\begin{aligned} M &:= M_+ \oplus M_- \\ \mathcal{D}(M) &:= \big(P_+ \mathcal{D}(\mathrm{Rot})\big) \oplus \big(P_- \mathcal{D}(\mathrm{Rot})\big) \end{aligned}$$

"Indestructible" Symmetry

$$\left. \begin{array}{r} C\, M_+ \, C = -M_- \\ M = M_+ \oplus M_- \end{array} \right\} \implies K\, M\, K = -M$$

has an **even particle-hole-type symmetry**

$$K := \sigma_1 \otimes C, \quad (\Psi_+, \Psi_-) \mapsto \big(\overline{\Psi_-}, \overline{\Psi_+}\big).$$

Fundamental Constituents

Complexified Maxwell Equations

1. *"Hamilton" operator* $M = \Big(W_+ \, \mathrm{Rot} \, \big\vert_{\mathrm{ran} \, P_+} \Big) \oplus \Big(W_- \, \mathrm{Rot} \, \big\vert_{\mathrm{ran} \, P_-} \Big)$
2. *Hilbert space*
   $\mathfrak{H} = \mathrm{ran} \, P_+ \oplus \mathrm{ran} \, P_- \subset L^2_{W_+}(\mathbb{R}^d, \mathbb{C}^6) \oplus L^2_{W_-}(\mathbb{R}^d, \mathbb{C}^6)$
3. Dynamics given by *Schrödinger equation*
   $$\mathrm{i}\partial_t \psi(t) = M \psi(t), \qquad \psi(0) = \big(P_+(\mathbf{E}, \mathbf{H}) \, , \, P_-(\mathbf{E}, \mathbf{H}) \big)$$
4. *Even particle-hole symmetry*: "Complex conjugation"
   $K = \sigma_1 \otimes C$

Reduction to Complex Fields with $\omega > 0$

Physically only **real states** relevant

$$\mathfrak{H}_{\mathbb{R}} := \Big\{ \big(\Psi_+ , \overline{\Psi_+} \big) \;\big\vert\; \Psi_+ \in \mathrm{ran}\, P_+ \Big\} \subset \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_-$$

$K \, M \, K = -M$ implies

$$\big(\mathbf{E}(t), \mathbf{H}(t) \big) = \mathrm{Re} \, \big(\mathrm{e}^{-\mathrm{i} t \, M_+} \Psi_+ \big) \simeq \mathrm{e}^{-\mathrm{i} t \, M} \, \mathrm{Re}_K \, \big(\Psi_+ , 0 \big)$$

where $\mathrm{Re}_K = \frac{1}{2} \big(\mathrm{id} + K \big)$ is the real part operator

$$\left. \begin{matrix} \text{Real transversal states} \\ (\mathbf{E}, \mathbf{H}) = \mathrm{Re} \, \Psi_+ \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Complex states with } \omega > 0 \\ \Psi_+ = P_+ (\mathbf{E}, \mathbf{H}) \end{matrix} \right.$$

$\mathrm{Re} = P_+^{-1} \Longrightarrow$ Study symmetries of M_+ (regular, $\pm$TR)

Reduction to Complex Fields with $\omega > 0$

Physically only **real states** relevant

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Fundamental Constituents

Reduced Description

1. *"Hamilton" operator* $M_+ = W_+ \, \mathrm{Rot} \big\vert_{\mathrm{ran}\, P_+}$
2. *Hilbert space* $\mathfrak{H}_+ = \mathrm{ran}\, P_+ \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$
3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H})$$

4. *Even particle-hole symmetry*: Implicit in construction

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathrm{Re}\, \Psi_+(t)$$

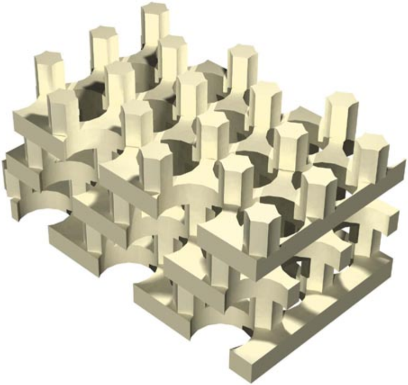
Back to the ray optics limit …

To Do List

We need to

1. clarify the precise **setting**,
2. prove the **existence of physical states**,
3. define the class of **observables** under consideration,
4. and *then* state the **ray optics limit**.

Assumptions on the Periodic Weights



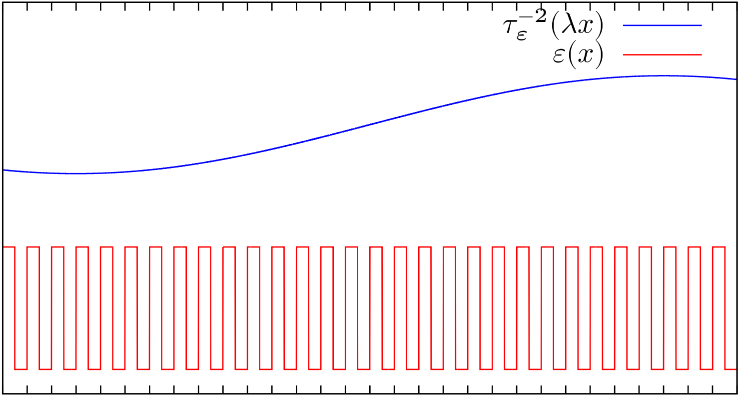
Johnson & Joannopoulos (2004)

Assumption (Periodic weights)

$$W_{+,0}(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

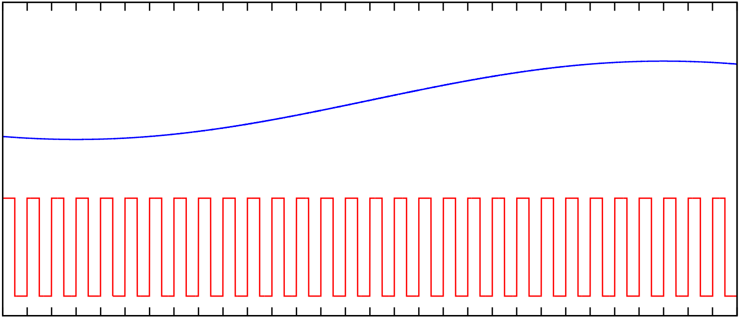
1. $W^*_{+,0} = W_{+,0}$ (lossless)
2. $0 < c\,\mathbf{1} \leq W_{+,0} \leq C\,\mathbf{1}$ (excludes metamaterials)
3. $W_{+,0}$ frequency-independent (no dispersion)
4. $W_{+,0}$ periodic

Macroscopic and Microscopic Degrees of Freedom



x [lattice constants]

Macroscopic and Microscopic Degrees of Freedom



$$\mathfrak{H}_0 \cong L^2(\mathcal{B}) \otimes L^2_{W_{+,0}}(\mathcal{W}) = \mathfrak{H}_{\text{macro}} \otimes \mathfrak{H}_{\text{micro}}$$

⇝ study macroscopic dynamics given a fixed microscopic state

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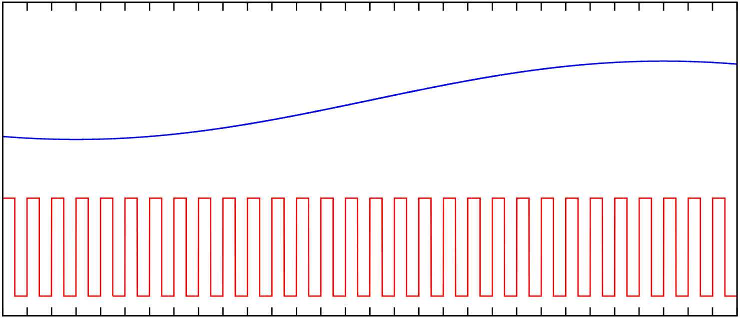

$\mathfrak{H}_0 \cong L^2(\mathcal{B}) \otimes L^2_{W_{+,0}}(\mathcal{W}) = \mathfrak{H}_{\text{macro}} \otimes \mathfrak{H}_{\text{micro}}$

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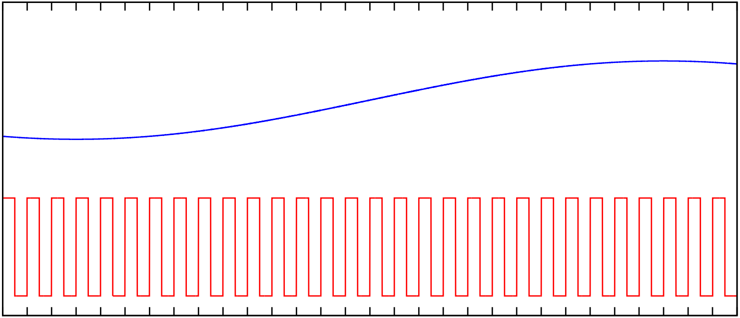

⇝ study macroscopic dynamics given a fixed microscopic state via space-adiabatic perturbation theory [PST (2002)]

Perturbations of Material Weights

Assumption (Slow modulations of material weights)

$$\begin{aligned} W_{+,\lambda}(x) &= \begin{pmatrix} \varepsilon_\lambda(x) & 0 \\ 0 & \mu_\lambda(x) \end{pmatrix} = S(\lambda x)^2 \, W_{+,0}(x) \\ &= \begin{pmatrix} \tau_\varepsilon^{-2}(\lambda x) & 0 \\ 0 & \tau_\mu^{-2}(\lambda x) \end{pmatrix} \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \end{aligned}$$

where $\tau_\varepsilon, \tau_\mu \in \mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^3, \mathbb{R})$, $\tau_\varepsilon, \tau_\mu \geq c > 0$

Slowly Modulated Maxwell Operator

Maxwell operator

- $M_\lambda := W_{+,\lambda} \, \mathrm{Rot} \, \big|_{\omega \geq 0}$
- $\mathfrak{H}_\lambda = J_\lambda \oplus G$ where

$$J_\lambda = \mathrm{ran} \, 1_{\{\omega > 0\}}(M_\lambda) = \mathrm{ran} \, P_{+,\lambda}$$

is the subspace of transversal fields and

$$G = \mathrm{ran} \, \nabla \oplus \nabla$$

is the subspace of longitudinal modes

- $\mathcal{D} := (J_\lambda \cap H^1) \oplus G$

$\leadsto$ Hilbert space depends on λ!

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Effective Dynamics

Goal

Approximate $\mathrm{e}^{-\mathrm{i}t M_\lambda}$ for **physical** states from a **narrow range of frequencies**, i. e. states which are

1. located in $J_\lambda = \mathrm{ran}\, 1_{\{\omega > 0\}}(M_\lambda)$ (subspace of transversal states) and
2. associated to specific frequency bands of M_0

up to higher-order errors in λ.

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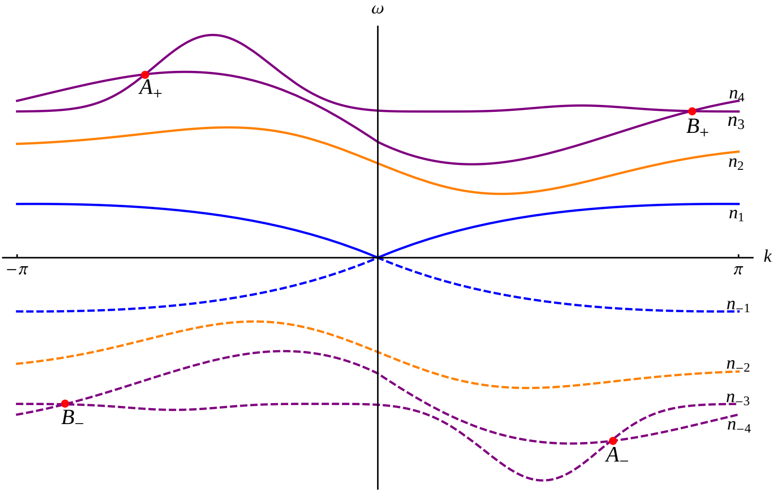

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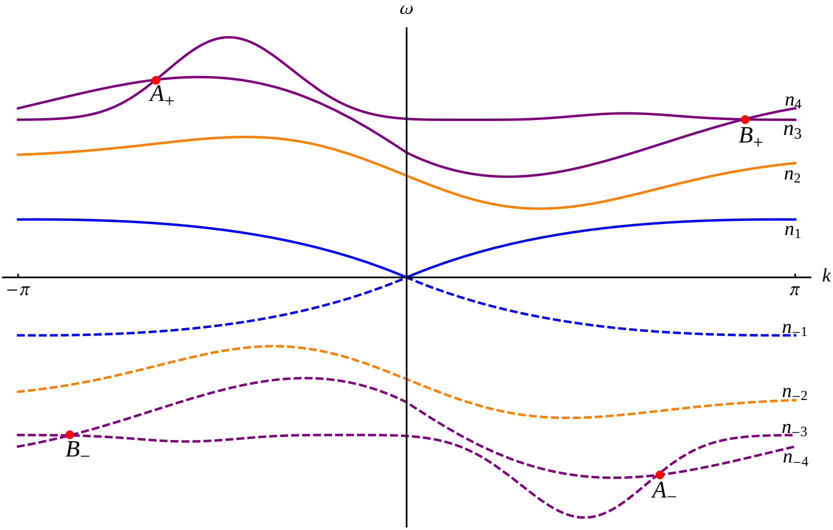
Effective *Semiclassical* Dynamics

Simplest case:
semiclassical dynamics
aka ray optics

Effective *Semiclassical* Dynamics



Effective *Semiclassical* Dynamics

Setup

- ω isolated, non-degenerate
- $\omega(k) \neq 0$ for all $k \in \mathbb{R}^3$ $\rightsquigarrow$ excludes ground state bands!
- Bloch function $k \mapsto \varphi(k)$
- Symbol of projection $(r,k) \mapsto \pi_0(r,k) := S^{-1}(r) \, |\varphi(k)\rangle\langle\varphi(k)| \, S(r)$ smooth
- Chern number associated to $k \mapsto |\varphi(k)\rangle\langle\varphi(k)|$ need not be zero! (then $k \mapsto \varphi(k)$ cannot be chosen purely real)

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Existence of Physical States

Theorem (De Nittis-L. 2012)

For any isolated family of bands there exists an orthogonal projection

$$\Pi_\lambda = \sum_{n=0}^{\infty} \lambda^n \, \mathsf{Op}_\lambda(\pi_n) + \mathcal{O}_{\|\cdot\|}(\lambda^\infty)$$

associated to an isolated family of bands so that up to $\mathcal{O}(\lambda^\infty)$

1. *states in its range are transversal,*
2. *it is a ΨDO,*
3. *the higher-order terms are computable (by recursion), and*
4. *its range is invariant under the dynamics, i. e. $[M_\lambda, \Pi_\lambda] = \mathcal{O}_{\|\cdot\|}(\lambda^\infty)$.*

Relevant Observables

Important difference between quantum mechanics and electromagnetism:

- *Quantum* mechanical observables $A = A^*$ are selfadjoint operators.
- *Electromagnetic* observables are *functionals of the fields,* $(\mathbf{E}, \mathbf{H}) \mapsto \mathcal{F}(\mathbf{E}, \mathbf{H}) \in \mathbb{C}$

We will only consider **quadratic observables** of the form

$$\mathcal{F}(\mathbf{E}, \mathbf{H}) = \big\langle (\mathbf{E}, \mathbf{H}), F\,(\mathbf{E}, \mathbf{H}) \big\rangle_{\mathfrak{H}_{+,\lambda} \oplus \mathfrak{H}_{-,\lambda}}, \quad F = F^* = \begin{pmatrix} F_{++} & F_{+-} \\ F_{-+} & F_{--} \end{pmatrix}$$

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Compatibility conditions

$$\begin{aligned} \mathcal{F}\big((\sigma_3 \otimes \mathbf{1})(\mathbf{E},\mathbf{H})\big) &= \mathcal{F}(\mathbf{E},\mathbf{H}) && \text{(no interaction $\pm$ frequencies)} \\ \mathcal{F}\big(K(\mathbf{E},\mathbf{H})\big) &= \overline{\mathcal{F}(\mathbf{E},\mathbf{H})} && \text{(reality condition)} \end{aligned}$$

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Compatibility conditions

$$\begin{aligned} \big[(\sigma_3 \otimes \mathbf{1}), F\big] = 0 \quad &\Longrightarrow \quad F_{+-} = 0 = F_{-+} \\ [K, F] = 0 \quad &\Longrightarrow \quad F_{--} = C\,F_{++}\,C \end{aligned}$$

Relevant Observables

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$$\mathcal{F}(\mathbf{E},\mathbf{H}) = \big\langle (\mathbf{E},\mathbf{H}) , F\,(\mathbf{E},\mathbf{H}) \big\rangle_{\mathfrak{H}_{+,\lambda} \oplus \mathfrak{H}_{-,\lambda}}$$

$$= 2\,\mathrm{Re}\,\Big\langle P_{+,\lambda}(\mathbf{E},\mathbf{H})\,,\,F_{++}\,P_{+,\lambda}(\mathbf{E},\mathbf{H}) \Big\rangle_{\mathfrak{H}_{\lambda}}$$

Definition of Relevant Class of Observables

Definition (Quadratic observables)

Suppose the electromagnetic observable

$$\begin{aligned}\mathcal{F}[(\mathbf{E},\mathbf{H})] &:= \mathbb{E}_f[(\mathbf{E},\mathbf{H})] \\ &:= 2\,\mathrm{Re}\,\Big\langle P_{+,\lambda}(\mathbf{E},\mathbf{H})\,,\,\mathrm{Op}_\lambda(f)\,P_{+,\lambda}(\mathbf{E},\mathbf{H})\Big\rangle_{\mathfrak{H}_\lambda}\end{aligned}$$

is defined in terms of a ΨDO associated to a selfadjoint-operator-valued function $f = f^*$.

1. We call $\mathcal{F}$ *scalar* if $f \equiv f \otimes \mathrm{id}_{L^2_{W_{+,0}}(\mathcal{W})}$ and $f \in \mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^6,\mathbb{C})$ are periodic in k.
2. We call $\mathcal{F}$ *non-scalar* if $f \in \mathcal{C}^\infty_{\mathrm{b}}\big(\mathbb{R}^6,\mathcal{B}(L^2_{W_{+,0}}(\mathcal{W}))\big)$ is an operator-valued function satisfying an equivariance condition.

Definition of Relevant Class of Observables

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Examples of quadratic observables

Local averages of

- the energy density,
- the Poynting vector, and
- components of the Maxwell stress tensor

The Ray Optics Limit

Dispersion relation

$$\Omega(r,k) = \tau_\varepsilon \, \tau_\mu \, \omega - \lambda \, \tau_\varepsilon \, \tau_\mu \, \mathcal{P} \cdot \nabla_r \ln \tfrac{\tau_\varepsilon}{\tau_\mu}$$

where

$$\mathcal{P}(k) := \mathrm{Im} \int_{\mathbb{T}^3} \mathrm{d}y \, \overline{\varphi^E(k,y)} \times \varphi^H(k,y).$$

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Equations of motion

$$\begin{pmatrix} \dot{r} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} -\lambda \, \Xi & +\mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \nabla_r \Omega \\ \nabla_k \Omega \end{pmatrix} ,$$

Berry curvature tensor $\Xi := \Big(\nabla_k \times \mathrm{i} \langle \varphi , \nabla_k \varphi \rangle_{L^2_{W_{+,0}}(\mathcal{W})} \Big)^{\times}$

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Theorem (Ray Optics for Scalar Quadratic Observables)

$$\mathcal{F}\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathbb{E}_{f \circ \Phi_t^\lambda}\big[(\mathbf{E}, \mathbf{H})\big] + \mathcal{O}(\lambda^2)$$

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Equations of motion

$$\begin{pmatrix} \dot{r} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & +\mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \nabla_r \Omega \\ \nabla_k \Omega \end{pmatrix},$$

Theorem (Ray Optics for Non-Scalar Quadratic Observables)

$$\mathcal{F}\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathbb{E}_{f_{\mathrm{ro}} \circ \Phi_t^\lambda}\big[(\mathbf{E}, \mathbf{H})\big] + \mathcal{O}(\lambda^2)$$

where $f_{\mathrm{ro}} = \pi_\lambda \sharp f \sharp \pi_\lambda + \mathcal{O}(\lambda^2)$

Semiclassics: Interpretation of Main Result

- First mathematically rigorous result
- Previously unknown $\mathcal{O}(\lambda)$ terms
  - $\mathcal{O}(\lambda)$ Term from symbol (change in field energy)
  - Rammal-Wilkinson-type term
  - Additional Berry curvature terms (geometric)
- Assumption $\omega_*(k) \neq 0 \; \forall k \in \mathbb{R}^3$ excludes ground state bands
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  - »Universal« behavior $\rightsquigarrow$ *free waves* with modified v_{light}
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Haldane & Raghu, Phys. Rev. A 78, 033834 (2008)

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$$\dot{k} = -\nabla_r \big(\tau_\varepsilon \, \tau_\mu \, \omega_* \big)$$

⇝ EOM are missing $\mathcal{O}(\lambda)$ terms, only leading-order correct

Semiclassics: Comparison to Notable Previous Results

Onoda, Murakami, Nagaosa, Phys. Rev. E 76, 066610 (2006)

- Use Sundaram–Niu variational technique + second quantization
- Semiclassical states $\Psi(r,k,z)$ parametrized by $(r,k) \in T^*\mathbb{R}^3$, $z \in S^2 \rightsquigarrow$ find extremals of functional

$$L = \Big\langle \Psi(r,k,z) \Big| \mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - M_\lambda^z \Big| \Psi(r,k,z) \Big\rangle$$

- Equations of motion:

$$\begin{aligned} \dot{r} &= +\nabla_k \big(\tau_\varepsilon \, \tau_\mu \, \omega_*\big) + \dot{k} \wedge \langle z | \widetilde{\Xi} | z \rangle + \text{other terms} \\ \dot{k} &= -\nabla_r \big(\tau_\varepsilon \, \tau_\mu \, \omega_*\big) + \text{other terms} \\ |\dot{z}\rangle &= \text{additional equation of motion} \end{aligned}$$

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- Equations of motion:

$$\begin{aligned}
\dot{r} &= +\nabla_k(\tau\omega_*) + \dot{k} \wedge \langle z|\widetilde{\Xi}|z\rangle + \\
&\qquad - \nabla_k\Big(\tau\omega_*\,\nabla_r \ln \tfrac{\tau_\varepsilon}{\tau_\mu} \cdot \langle z|\tfrac{1}{2}(\widetilde{\mathcal{A}}^E - \widetilde{\mathcal{A}}^H)|z\rangle\Big) + \\
&\qquad + \mathrm{i}\langle z|\big[(\tau\omega_*\,\nabla_r \ln \tfrac{\tau_\varepsilon}{\tau_\mu}) \cdot \tfrac{1}{2}(\widetilde{\mathcal{A}}^E - \widetilde{\mathcal{A}}^H)\,,\,\tfrac{1}{2}(\widetilde{\mathcal{A}}^E + \widetilde{\mathcal{A}}^H)\big]|z\rangle \\
\dot{k} &= -\nabla_r(\tau\omega_*) - \nabla_r\Big(\tau\omega_*\,\nabla_r \ln \tfrac{\tau_\varepsilon}{\tau_\mu} \cdot \langle z|\tfrac{1}{2}(\widetilde{\mathcal{A}}^E - \widetilde{\mathcal{A}}^H)|z\rangle\Big) \\
|\dot{z}\rangle &= \mathrm{i}\Big(-\dot{k} \cdot \tfrac{1}{2}(\widetilde{\mathcal{A}}^E + \widetilde{\mathcal{A}}^H) + \tau\omega_*\,\nabla_r \ln \tfrac{\tau_\varepsilon}{\tau_\mu} \cdot \tfrac{1}{2}(\widetilde{\mathcal{A}}^E - \widetilde{\mathcal{A}}^H)\Big)|z\rangle
\end{aligned}$$

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- Result is not readily comparable to ours

Quantum Analogies Investigated in the Past

- Schrödinger formalism of Maxwell equations
  Physics: *in vacuo* $\rightsquigarrow$ Dirac, Wigner, ... (1920s)
  Mathematics: *non-gyrotropic* $\rightsquigarrow$ Birman & Solomyak (1987)
- Random Maxwell & acoustic operators
  Figotin & Klein (1997)
- Derivation of non-linear Schrödinger equation from non-linear Maxwell equations
  Babin & Figotin (early 2000s)
- Adiabatic perturbation theory for photonic crystals
  De Nittis & L. (2014)
- Ray optics in photonics $\longleftrightarrow$ semiclassics in quantum mechanics
  De Nittis & L. (2015) for photonic crystals
- Classification of Photonic Topological Insulators
  De Nittis & L. (2014 & 2016)

Open Problems

For operators of product form

$$M = W\,D$$

- Scattering theory
  $\rightsquigarrow$ technical conditions on W and D?
- What if $W \not> 0$ (e. g. in metals or for magnons)
  $\rightsquigarrow$ Theory of Krein spaces
- *Non-linear* topological insulators (e. g. in photonic or magnonic crystals)
  $\rightsquigarrow$ Existence of topological solitons?
- Dispersion
- Spectral problems
  e. g. M periodic Maxwell operator, $W \in L^\infty$
  $\Longrightarrow \sigma(M) \setminus \{0\} = \sigma_{\mathrm{ac}}(M) \setminus \{0\}$

Thank you for your attention!