UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Solutions 1
(2014.09.12)

Quantum Mechanics for Spin Systems & the Uncertainty Principle

Homework Problems

1. The Pauli matrices

Consider the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(i) Prove $\sigma_j \, \sigma_k = \delta_{jk} \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \, \sigma_l$ where ϵ_{jkl} is the epsilon tensor.

(ii) Prove that any 2×2 matrix can be written as the linear combination of the identity and the three Pauli matrices with coefficients h_0 and $h = (h_1, h_2, h_3)$,

$$\mathrm{Mat}_{\mathbb{C}}(2) \ni A = (a_{jk})_{1 \leq j,k \leq 2} = h_0 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^{3} h_j \, \sigma_j =: h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma. \tag{1}$$

Hint: Use that $\mathrm{Mat}_{\mathbb{C}}(2)$ is finite-dimensional.

(iii) Now assume that the coefficients $h_0, \ldots, h_3$ in equation (1) are real. Show that then the resulting matrix $H = h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma$ is hermitian. Compute the eigenvalues $E_{\pm}(h_0, h)$ of H in terms of the coefficients h_0 and h.

(iv) Use (i) to prove that for real $h_0, \ldots, h_3$

$$P_{\pm}(h_0, h) = \frac{1}{2} \left(\mathrm{id}_{\mathbb{C}^2} \pm \frac{h \cdot \sigma}{|h|} \right) , \qquad h \neq 0 \in \mathbb{R}^3, \; |h| := \sqrt{h_1^2 + h_2^2 + h_3^2},$$

are the projections onto the eigenspaces for the two eigenvalues $E_{\pm}(h_0, h)$ of H.

(v) Compute the trace of H.

Note: In physics especially, one frequently writes $h \cdot \sigma$ for $\sum_{j=1}^{3} h_j \, \sigma_j$ where $h = (h_1, h_2, h_3)$.

Solution:

(i) This follows from direct computation: for $j = k$ we obtain

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathrm{id}_{\mathbb{C}^2}$$

while for $j < k$

$$\begin{aligned}
\sigma_1\,\sigma_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} +\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i}\,\sigma_3 \\
\sigma_1\,\sigma_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\,\sigma_2 \\
\sigma_2\,\sigma_3 &= \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & +\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\,\sigma_1
\end{aligned}$$

In other words, we have shown (i) for $j < k$.

To show (i) in the remaining cases, we use that the $\sigma_j = \sigma_j^*$ are hermitian matrices, and hence for $j < k$ we obtain

$$\begin{aligned}
\sigma_k\,\sigma_j = \bigl(\sigma_j\,\sigma_k\bigr)^* &= \left(\delta_{jk}\,\mathrm{id}_{\mathbb{C}^2} + \mathrm{i}\sum_{l=1}^{3} \epsilon_{jkl}\,\sigma_l\right)^* \\
&= \delta_{jk}\,\mathrm{id}_{\mathbb{C}^2} - \mathrm{i}\sum_{l=1}^{3} \epsilon_{jkl}\,\sigma_l = \delta_{jk}\,\mathrm{id}_{\mathbb{C}^2} + \mathrm{i}\sum_{l=1}^{3} \epsilon_{kjl}\,\sigma_l .
\end{aligned}$$

This proves (i).

(ii) The vector space of 2×2 matrices is four-dimensional, $\dim \mathrm{Mat}_{\mathbb{C}}(2) = 4$, and seeing as the 4 vectors $\bigl\{\mathrm{id}_{\mathbb{C}^2}, \sigma_1, \sigma_2, \sigma_3\bigr\}$ are linearly independent, they form a basis of $\mathrm{Mat}_{\mathbb{C}}(2)$.

(iii) In case $h_0, \ldots, h_3$ are real,

$$\begin{aligned}
H^* &= \bigl(h_0\,\mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma\bigr)^* = \overline{h_0}\,\mathrm{id}_{\mathbb{C}^2} + \overline{h} \cdot \sigma \\
&= h_0\,\mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma = H
\end{aligned}$$

is hermitian and we can compute both eigenvalues: the characteristic polynomial of H is

$$\begin{aligned}
\chi(\lambda) = \det\bigl(\lambda\,\mathrm{id}_{\mathbb{C}^2} - H\bigr) &= \det \begin{pmatrix} \lambda - h_0 - h_3 & h_1 - \mathrm{i}\,h_2 \\ h_1 + \mathrm{i}\,h_2 & \lambda - h_0 + h_3 \end{pmatrix} \\
&= \bigl((\lambda - h_0) - h_3\bigr)\,\bigl((\lambda - h_0) + h_3\bigr) - \bigl(h_1 - \mathrm{i}h_2\bigr)\bigl(h_1 + \mathrm{i}\,h_2\bigr) \\
&= (\lambda - h_0)^2 - \bigl(h_1^2 + h_2^2 + h_3^2\bigr) = (\lambda - h_0)^2 - |h|^2 ,
\end{aligned}$$

and hence, the eigenvalues are $E_\pm(h_0, h) = h_0 \pm |h|$.

(iv) The product

$$H\,P_\pm = \bigl(h_0\,\mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma\bigr)\,P_\pm = h_0\,P_\pm + \frac{1}{2}\left(h \cdot \sigma \pm \frac{(h \cdot \sigma)^2}{|h|}\right)$$

involves the square of $h \cdot \sigma$ which can be computed with the help of (i):

$$\begin{aligned}
\bigl(h \cdot \sigma\bigr)^2 &= \sum_{j,k=1}^{3} h_j \, h_k \, \sigma_j \, \sigma_k \\
&= \sum_{j=1}^{3} h_j^2 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{\substack{j,k,l=1,2,3 \\ j \neq k}} h_j \, h_k \, \mathrm{i} \, \epsilon_{jkl} \, \sigma_l \\
&= |h|^2 \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \Biggl(\sum_{\substack{j,k=1,2,3 \\ j \neq k}} h_j \, h_k \, \mathrm{i} \, \epsilon_{jkl} \Biggr) \, \sigma_l = |h|^2 \, \mathrm{id}_{\mathbb{C}^2}
\end{aligned}$$

Hence, we can factor out $E_\pm$ and obtain (iv):

$$\begin{aligned}
H \, P_\pm &= h_0 \, P_\pm + \frac{1}{2} \left(h \cdot \sigma \pm \frac{|h|^2 \, \mathrm{id}_{\mathbb{C}^2}}{|h|} \right) \\
&= h_0 \, P_\pm \pm |h| \, \frac{1}{2} \left(\mathrm{id}_{\mathbb{C}^2} \pm \frac{h \cdot \sigma}{|h|} \right) \\
&= \bigl(h_0 \pm |h| \bigr) \, P_\pm = E_\pm \, P_\pm
\end{aligned}$$

(v) The trace is just the sum over the diagonal elements of the matrices, and clearly, the Pauli matrices are all traceless. Hence, we compute

$$\begin{aligned}
\mathrm{tr} H &= \mathrm{tr}\bigl(h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma \bigr) \\
&= h_0 \, \mathrm{tr} \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^{3} h_j \, \mathrm{tr} \, \sigma_j = 2 h_0 .
\end{aligned}$$

2. Functional calculus for 2×2 matrices

Let f be a piecewise continuous function and $H = H^*$ a hermitian 2×2 matrix. Then define

$$f(H) := \sum_{j=\pm} f(E_\pm)\, P_\pm \tag{2}$$

where $E_\pm$ are the eigenvalues of H and $P_\pm$ the two projections from problem 1.

(i) Compute $f(H)$ defined as in equation (2) for $H = h \cdot \sigma$, $h \neq 0$, and

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

(ii) Show that $f(H)$ for $f(x) = \mathrm{e}^{-\mathrm{i}tx}$ (defined via (2)) coincides with the matrix exponential, i. e.

$$f(H) = \mathrm{e}^{-\mathrm{i}th_0} \left(\cos\bigl(|h|\, t\bigr) - \frac{\mathrm{i}}{|h|} \sin\bigl(|h|\, t\bigr)\, h \cdot \sigma \right) = \mathrm{e}^{-\mathrm{i}tH} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!}\, H^n. \tag{3}$$

Hint: Use $\mathrm{e}^{-\mathrm{i}t(h_0 + h\cdot\sigma)} = \mathrm{e}^{-\mathrm{i}th_0}\, \mathrm{e}^{-\mathrm{i}th\cdot\sigma}$.

(iii) Assuming h_0, h_1, h_2, h_3 are real, compute $\psi(t)$ for the initial condition $\psi(0) = \psi_0 \in \mathbb{C}^2$:

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \bigl(h_2\, \sigma_2 + h_3\, \sigma_3\bigr)\psi(t)$

(b) $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = h_2\, \sigma_2 \psi(t)$

(c) $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \bigl(h_0\, \mathrm{id}_{\mathbb{C}^2} + h_3\, \sigma_3\bigr)\psi(t)$

Solution:

(i) $f(H) = f(|h|)\, P_+ + f(-\,|h|)\, P_- = P_+$

(ii) For $h = 0$, H is a scalar multiple of the identity matrix and equation (3) holds. So let us assume $h \neq 0$. Then we first compute the left-hand side:

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}tx}(H) &= \mathrm{e}^{-\mathrm{i}t(h_0+|h|)}\, P_+ + \mathrm{e}^{-\mathrm{i}t(h_0-|h|)}\, P_- \\
&= \frac{1}{2}\Bigl(\mathrm{e}^{-\mathrm{i}t(h_0+|h|)} + \mathrm{e}^{-\mathrm{i}t(h_0-|h|)}\Bigr)\, \mathrm{id}_{\mathbb{C}^2} + \frac{1}{2}\Bigl(\mathrm{e}^{-\mathrm{i}t(h_0+|h|)} - \mathrm{e}^{-\mathrm{i}t(h_0-|h|)}\Bigr)\, \frac{h\cdot\sigma}{|h|} \\
&= \mathrm{e}^{-\mathrm{i}th_0} \left(\cos\bigl(|h|\, t\bigr) - \frac{\mathrm{i}}{|h|} \sin\bigl(|h|\, t\bigr)\, h \cdot \sigma \right)
\end{aligned}$$

To obtain the right-hand side, we note

$$\begin{aligned}
\bigl(h\cdot\sigma\bigr)^2 &= \sum_{j,k=1,2,3} h_j\, h_k\, \sigma_j\, \sigma_k = \sum_{j=1,2,3} h_j^2\, \sigma_j^2 + \sum_{\substack{j,k=1,2,3 \\ j\neq k}} h_j\, h_k\, \sigma_j\, \sigma_k \\
&= \sum_{j=1,2,3} h_j^2\, \mathrm{id}_{\mathbb{C}^2} + \sum_{l=1,2,3} \sum_{\substack{j,k=1,2,3 \\ j\neq k}} h_j\, h_k\, \epsilon_{jkl}\, \sigma_l = h^2,
\end{aligned}$$

and thus we identify a pattern in $\bigl(h\cdot\sigma\bigr)^n$:

$$\begin{aligned}
\bigl(h\cdot\sigma\bigr)^{2n} &= |h|^{2n}\, \mathrm{id}_{\mathbb{C}^2} \\
\bigl(h\cdot\sigma\bigr)^{2n+1} &= \bigl(h\cdot\sigma\bigr)^{2n}\, h\cdot\sigma = |h|^{2n}\, h\cdot\sigma
\end{aligned}$$

This means that we can compute the matrix exponential after splitting the sum into even and odd terms:

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}tH} &= \mathrm{e}^{-\mathrm{i}th_0}\,\mathrm{e}^{-\mathrm{i}th\cdot\sigma} = \mathrm{e}^{-\mathrm{i}th_0}\,\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}\big(h\cdot\sigma\big)^n \\
&= \mathrm{e}^{-\mathrm{i}th_0}\,\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^{2n}}{(2n)!}\big(h\cdot\sigma\big)^{2n} + \mathrm{e}^{-\mathrm{i}th_0}\,\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^{2n+1}}{(2n+1)!}\big(h\cdot\sigma\big)^{2n+1} \\
&= \mathrm{e}^{-\mathrm{i}th_0}\,\sum_{n=0}^{\infty}(-1)^n\frac{(|h|\,t)^{2n}}{(2n)!}\,\mathrm{id}_{\mathbb{C}^2} - \frac{\mathrm{i}}{|h|}\,\mathrm{e}^{-\mathrm{i}th_0}\,\sum_{n=0}^{\infty}(-1)^n\frac{(|h|\,t)^{2n+1}}{(2n+1)!}\,h\cdot\sigma \\
&= \mathrm{e}^{-\mathrm{i}th_0}\,\Big(\cos\big(|h|\,t\big)\,\mathrm{id}_{\mathbb{C}^2} - \frac{\mathrm{i}}{|h|}\,\sin\big(|h|\,t\big)\,h\cdot\sigma\Big)
\end{aligned}$$

Thus, left- and right-hand side agree.

(iii) (a)

$$\begin{aligned}
\psi(t) &= \mathrm{e}^{tH}\psi_0 = \mathrm{e}^{t|h|}\,P_+(0,0,h_2,h_3) + \mathrm{e}^{-t|h|}\,P_-(0,0,h_2,h_3) \\
&= \frac{1}{2}\big(\mathrm{e}^{+t|h|} + \mathrm{e}^{-t|h|}\big)\,\psi_0 + \frac{1}{2}\frac{\mathrm{e}^{+t|h|} - \mathrm{e}^{-t|h|}}{|h|}\,\big(h_2\,\sigma_2 + h_3\,\sigma_3\big)\psi_0 \\
&= \cosh\big(t\,|h|\big)\,\psi_0 + \sinh\big(t\,|h|\big)\,\big(h_2\,\sigma_2 + h_3\,\sigma_3\big)\psi_0
\end{aligned}$$

(b) $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0 = \cos\big(t|h_2|\big)\,\psi_0 - \mathrm{i}\,\frac{h_2}{|h_2|}\,\sin\big(t|h_2|\big)\,\sigma_2\psi_0$

(c) $\psi(t) = \mathrm{e}^{+\mathrm{i}tH}\psi_0 = \mathrm{e}^{+\mathrm{i}th_0}\,\cos\big(t|h_2|\big)\,\psi_0 + \mathrm{i}\,\mathrm{e}^{+\mathrm{i}th_0}\,\frac{h_3}{|h_3|}\,\sin\big(t|h_3|\big)\,\sigma_3\psi_0$

3. Uncertainty of Gauß functions

Compute the right-hand side of Heisenberg's uncertainty principle

$$\sigma_\psi(\hat{x})\,\sigma_\psi(-\mathrm{i}\hbar\partial_x)$$

in one dimension for

(i) $\psi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}}\mathrm{e}^{-\frac{\lambda}{2}x^2}$, $\lambda > 0$, and

(ii) $\varphi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}}\mathrm{e}^{+\mathrm{i}x\xi_0}\,\mathrm{e}^{-\frac{\lambda}{2}(x-x_0)^2}$, $\lambda > 0, x_0, \xi_0 \in \mathbb{R}$.

Here, the standard deviation

$$\sigma_\psi(H) := \sqrt{\mathbb{E}_\psi\Big(\big(H - \mathbb{E}_\psi(H)\big)^2\Big)}$$

for a selfadjoint operator $H = H^*$ with respect to ψ, $\|\psi\| = 1$, is defined as in the lecture notes via the expectation value

$$\mathbb{E}_\psi(H) := \langle \psi, H\psi \rangle.$$

Solution:

(i) First of all, since $\psi(-x) = \psi(x)$ the expectation value

$$\mathbb{E}_{\psi_\lambda}(\hat{x}) = \int_{\mathbb{R}} \mathrm{d}x\,\sqrt{\frac{\lambda}{\pi}}\,x\,\mathrm{e}^{-\lambda x^2} = 0$$

necessarily vanishes. Similarly,

$$\begin{aligned}\mathbb{E}_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x) &= -\mathrm{i}\hbar\,\big\langle \psi_\lambda, \partial_x\psi_\lambda\big\rangle = -\mathrm{i}\hbar\,\big\langle \mathcal{F}\psi_\lambda, \mathcal{F}\partial_x\psi_\lambda\big\rangle \\ &= \hbar\,\big\langle \psi_{1/\lambda}, \hat{\xi}\psi_{1/\lambda}\big\rangle = 0\end{aligned}$$

is also 0, because the Fourier transform of a Gaußian is also a Gaußian.

That means we can compute the first standard deviation by partial integration:

$$\begin{aligned}\sigma_{\psi_\lambda}(\hat{x})^2 &= \mathbb{E}_{\psi_\lambda}\Big(\big(\hat{x} - \mathbb{E}_{\psi_\lambda}(\hat{x})\big)^2\Big) = \mathbb{E}_{\psi_\lambda}(\hat{x}^2) = \sqrt{\frac{\lambda}{\pi}}\int_{\mathbb{R}}\mathrm{d}x\,x^2\,\mathrm{e}^{-\lambda x^2} \\ &= \frac{1}{\lambda\sqrt{\pi}}\int_{\mathbb{R}}\mathrm{d}x\,x^2\,\mathrm{e}^{-x^2} = \left[-\frac{1}{2\lambda\sqrt{\pi}}\,x\,\mathrm{e}^{-x^2}\right]_{-\infty}^{+\infty} + \frac{1}{2\lambda\sqrt{\pi}}\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{-x^2} = \frac{1}{2\lambda}\end{aligned}$$

To compute the other standard deviation, we note that since the Fourier transform of a Gaußian is a Gaußian with inverse width,

$$(\mathcal{F}\psi_\lambda)(\xi) = \sqrt[4]{\frac{\lambda}{\pi}}\,\big(\mathcal{F}\mathrm{e}^{-\frac{\lambda}{2}x^2}\big)(\xi) = \frac{1}{\sqrt[4]{\lambda\pi}}\mathrm{e}^{-\frac{1}{2\lambda}\xi^2} = \psi_{1/\lambda},$$

we can relate $\sigma_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x)$ to $\sigma_{\psi_\lambda}(\hat{x})$,

$$\begin{aligned}\sigma_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x)^2 &= -\hbar^2\,\big\langle \psi_\lambda, \partial_x^2\psi_\lambda\big\rangle = -\hbar^2\,\big\langle \mathcal{F}\psi_\lambda, \mathcal{F}\partial_x^2\psi_\lambda\big\rangle \\ &= +\hbar^2\,\big\langle \mathcal{F}\psi_{1/\lambda}, \hat{\xi}^2\psi_{1/\lambda}\big\rangle = \frac{\hbar^2}{2\lambda^{-1}}.\end{aligned}$$

Hence, ψ_λ minimizes the uncertainty relation,

$$\sigma_{\psi_\lambda}(\hat{x})\;\sigma_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x) = \frac{1}{\sqrt{2\lambda}}\sqrt{\frac{\hbar^2\lambda}{2}} = \frac{\hbar}{2} \geq \frac{\hbar}{2}.$$

(ii) We will reuse the results from (i) as much as possible: the mean of φ_λ is x_0:

$$\begin{aligned}\mathbb{E}_{\varphi_\lambda}(\hat{x}) &= \int_{\mathbb{R}} \mathrm{d}x \, x \, \bigl| \mathrm{e}^{+\mathrm{i}x\xi_0} \, \psi_\lambda(x - x_0) \bigr|^2 = \int_{\mathbb{R}} \mathrm{d}x \, (x + x_0) \, \bigl|\psi_\lambda(x)\bigr|^2 \\ &= x_0 \, \bigl\|\psi_\lambda(x)\bigr\|^2 = x_0\end{aligned}$$

Hence, the standard deviation of φ_λ coincides with that of ψ_λ:

$$\begin{aligned}\sigma_{\varphi_\lambda}(\hat{x})^2 = \mathbb{E}_{\varphi_\lambda}\bigl((\hat{x} - x_0)^2\bigr) &= \int_{\mathbb{R}} \mathrm{d}x \, (x - x_0)^2 \, \bigl| \mathrm{e}^{+\mathrm{i}x\xi_0} \, \psi_\lambda(x - x_0) \bigr|^2 \\ &= \int_{\mathbb{R}} \mathrm{d}x \, x^2 \, \bigl|\psi_\lambda(x)\bigr|^2 = \frac{1}{2\lambda}\end{aligned}$$

Since the Fourier transform intertwines taking derivatives with multiplying by monomials and maps Gaußians on Gaußians of inverse width,

$$(\mathcal{F}\varphi_\lambda)(\xi) = \bigl(\mathcal{F}\mathrm{e}^{+\mathrm{i}x\xi_0} \, \psi_\lambda(\,\cdot\, - x_0)\bigr)(\xi) = \bigl(\mathcal{F}\psi_\lambda(\,\cdot\, - x_0)\bigr)(\xi - \xi_0) = \mathrm{e}^{-\mathrm{i}\xi x_0} \, \psi_{1/\lambda}(\xi - \xi_0),$$

we obtain the same integral (up to $\hbar^2$) where λ is replaced by λ^{-1},

$$\sigma_{\varphi_\lambda}(-\mathrm{i}\hbar\partial_x)^2 = \frac{\hbar^2\lambda}{2}.$$

Hence, also shifted Gaußians have minimal uncertainty,

$$\sigma_{\varphi_\lambda}(\hat{x}) \, \sigma_{\varphi_\lambda}(-\mathrm{i}\hbar\partial_x) = \frac{\hbar}{2} \geq \frac{\hbar}{2}.$$

4. The framework of physical theories (optional)

Identify (1) states, (2) observables and (3) dynamical equations for the following physical theories:

(i) Classical mechanics on $\mathbb{R}^d$

(ii) Classical electromagnetism

Solution:

(i) (1) *States:* Probability measures on phase space $\mathbb{R}^{2d}$

(2) *Observables:* Smooth functions on phase space $\mathbb{R}^{2d}$

(3) *Dynamical equations:* Given a hamiltonian (energy function) h, we can either propose Hamilton's equations of motion,

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} +\nabla_p h \\ -\nabla_q h \end{pmatrix},$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \big\{h, f(t)\big\}, \qquad f(0) = f,$$

where $\{f, g\} = \nabla_p f \cdot \nabla_q g - \nabla_q f \cdot \nabla_p g$ is the Poisson bracket

(ii) (1) *States:* Electromagnetic fields $(\mathbf{E}, \mathbf{B})$, i. e. vector fields on $\mathbb{R}^3$ which satisfy the two source Maxwell equation:

$$\nabla \cdot \mathbf{E} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$

where ρ is a charge density

(2) *Observables:* Functionals $\mathcal{F}(\mathbf{E}, \mathbf{B}) \in \mathbb{R}$ on the fields

(3) *Dynamical equations:* the dynamical Maxwell equations

$$\partial_t \mathbf{E} = +\nabla \wedge \mathbf{B} - j$$
$$\partial_t \mathbf{B} = -\nabla \wedge \mathbf{E}$$

where j is the current density