UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 1
(2013.09.10)

Classification of Differential Equations & Solution to the Exponential Equation

Homework Problems

1. Classification of differential equation (7 points)

Classify the following differential equations: are they ODEs ore PDEs, linear homogeneous, linear-inhomogeneous or non-linear?

(i) $\mathrm{i}\partial_t u = -\partial_x^2 u + V u$ (V is a real-valued function)

(ii) $\partial_t\big(u^2\big) = u$

(iii) $\partial_t u = \partial_x^2 u - V u + f(t)$ (V and $f(t)$ are real-valued functions)

(iv) $\mathrm{i}\partial_t u = -\partial_x^2 u + |u|^2\, u$

(v) $\partial_x u = 0$ where u is a function of x and t

(vi) $\partial_t u + u\,\partial_x u + \partial_x^3 u = 0$

(vii) $\partial_t u + \partial_x\big(u^2\big) = 0$

Solution:

(i) PDE, linear-homogeneous (Schrödinger equation) [1]

(ii) ODE, non-linear [1]

(iii) PDE, linear-inhomogeneous (heat equation with source) [1]

(iv) PDE, non-linear (non-linear Schrödinger equation) [1]

(v) ODE, linear-homogeneous (conservation law) [1]

(vi) PDE, non-linear (Korteveg–de Vries equation) [1]

(vii) PDE, non-linear (Hamilton-Jacobi-type equation) [1]

2. The matrix-valued exponential equation (15 points)

Let H be a $n \times n$ matrix with complex entries and define the matrix exponential

$$\mathrm{e}^{tH} := \sum_{k=0}^{\infty} \frac{t^k}{k!} H^k.$$

(We set $H^0 := \mathrm{id}_{\mathbb{C}^n}$ to the $n \times n$ identity matrix.) In (i)–(iii), show that $x(t) = \mathrm{e}^{tH} x_0$ solves

$$\dot{x}(t) = Hx(t), \qquad x(0) = x_0 \in \mathbb{C}^n. \tag{1}$$

(i) Show that $\mathrm{e}^{t_1 H}\,\mathrm{e}^{t_2 H} = \mathrm{e}^{(t_1+t_2)H}$ holds for all $t_1, t_2 \in \mathbb{R}$.

(ii) Prove that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tH} = H\,\mathrm{e}^{tH}$ using the definition of the derivative as a limit. (You may interchange limits and infinite sums without proof.)

(iii) Show that $\Phi_t := \mathrm{e}^{tH}$ is the flow associated to the ODE (1).

Solution:

(i) Using

$$(t_1+t_2)^m = \sum_{k=0}^{m} \binom{m}{k} t_1^k\, t_2^{m-k} \qquad [1]$$

we obtain

$$\begin{aligned}
\mathrm{e}^{t_1 H}\,\mathrm{e}^{t_2 H} &\overset{[1]}{=} \left(\sum_{k=0}^{\infty} \frac{t_1^k}{k!} H^k\right)\left(\sum_{j=0}^{\infty} \frac{t_2^j}{j!} H^j\right) \overset{[1]}{=} \sum_{m=0}^{\infty} \sum_{j+k=m} \frac{1}{k!}\frac{1}{j!}\, t_1^k\, t_2^j\, H^{k+j} \\
&= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \underbrace{\frac{1}{k!(m-k)!}}_{=\frac{1}{m!}\binom{m}{k}} t_1^k\, t_2^{m-k}\right) H^m \overset{[1]}{=} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\sum_{k=0}^{m} \binom{m}{k} t_1^k\, t_2^{m-k}\right) H^m \\
&\overset{[1]}{=} \sum_{m=0}^{\infty} \frac{1}{m!}\,(t_1+t_2)^m\, H^m \overset{[1]}{=} \mathrm{e}^{(t_1+t_2)H}.
\end{aligned}$$

(ii) We use the definition of the derivative as a limit:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tH} &\overset{[1]}{=} \lim_{\delta\to 0} \frac{1}{\delta}\Big(\mathrm{e}^{(t+\delta)H} - \mathrm{e}^{tH}\Big) \overset{[1]}{=} \lim_{\delta\to 0} \frac{1}{\delta}\Big(\mathrm{e}^{\delta H} - \mathrm{id}_{\mathbb{C}^n}\Big)\,\mathrm{e}^{tH} \\
&= \lim_{\delta\to 0}\left(\sum_{k=1}^{\infty} \frac{\delta^{k-1}}{k!} H^k\right)\mathrm{e}^{tH} \overset{[1]}{=} \left(H + \lim_{\delta\to 0}\sum_{k=2}^{\infty} \frac{\delta^{k-1}}{k!} H^k\right)\mathrm{e}^{tH} \\
&\overset{[1]}{=} \left(H + \sum_{k=2}^{\infty} \lim_{\delta\to 0}\frac{\delta^{k-1}}{k!} H^k\right)\mathrm{e}^{tH} \overset{[1]}{=} H\,\mathrm{e}^{tH}.
\end{aligned}$$

(iii) First of all, by (ii) we know $x(t) = \mathrm{e}^{tH} x_0$ solves (1) with initial condition $x(0) = x_0$. [1] Now we need to verify the three group properties flows possess:

(1) $\Phi_0 = \mathrm{e}^{0H} = \sum_{k=0}^{\infty} \frac{0^k}{k!} H^k = \mathrm{id}_{\mathbb{C}^n}$ [1]

(2) $\Phi_{t_1} \circ \Phi_{t_2} = \mathrm{e}^{t_1 H}\,\mathrm{e}^{t_2 H} \overset{(ii)}{=} \mathrm{e}^{(t_1+t_2)H} = \Phi_{t_1+t_2}$ [1]

(3) $\Phi_t \circ \Phi_{-t} \overset{(2)}{=} \Phi_{t-t} \overset{(1)}{=} \mathrm{id}_{\mathbb{C}^n}$ [1]

3. Dynamics of a classical spin (14 points)

Consider the equation

$$\begin{pmatrix} \dot n_1 \\ \dot n_2 \\ \dot n_3 \end{pmatrix} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \qquad n(0) = n^{(0)} \in \mathbb{R}^3,$$

where $a \times b$ is the usual crossed product of vectors in $\mathbb{R}^3$ and $\omega \in \mathbb{R}$.

(i) Compute the flow.

(ii) Find the solution $n(t)$ for the particular initial condition $n^{(0)} = (1, 0, 0)$ *without* explicitly computing the matrix exponential. (Hint: Work smart, not hard.)

(iii) Show $\lVert n(t) \rVert = \lVert n^{(0)} \rVert$, i. e. verify that the length of the spin vector $\lVert n \rVert := \sqrt{n \cdot n} = \sqrt{n_1^2 + n_2^2 + n_3^2}$ is conserved.

Solution:

(i) The cross product can also be expressed as a matrix:

$$\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega\, n_3 \\ +\omega\, n_2 \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & +\omega & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} =: Hn$$

We can explicitly diagonalize the matrix

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & +\omega & 0 \end{pmatrix} \overset{[3]}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & +\mathrm{i}\,\omega & 0 \\ 0 & 0 & -\mathrm{i}\,\omega \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{pmatrix}^*$$

and thus, the matrix exponential is

$$\begin{aligned} \mathrm{e}^{tH} &\overset{[1]}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{pmatrix} \exp\left(t \begin{pmatrix} 0 & 0 & 0 \\ 0 & +\mathrm{i}\,\omega & 0 \\ 0 & 0 & -\mathrm{i}\,\omega \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{pmatrix}^* \\ &\overset{[1]}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^0 & 0 & 0 \\ 0 & \mathrm{e}^{+\mathrm{i}\,\omega t} & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\,\omega t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & \mathrm{i}/\sqrt{2} \\ 0 & 1/\sqrt{2} & -\mathrm{i}/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\mathrm{e}^{+\mathrm{i}\,\omega t}}{\sqrt{2}} & \frac{\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\,\omega t}}{\sqrt{2}} \\ 0 & \frac{\mathrm{e}^{-\mathrm{i}\,\omega t}}{\sqrt{2}} & -\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\omega t}}{\sqrt{2}} \end{pmatrix} \\ &\overset{[1]}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\left(\mathrm{e}^{+\mathrm{i}\,\omega t} + \mathrm{e}^{-\mathrm{i}\,\omega t}\right) & -\frac{1}{\mathrm{i}2}\left(\mathrm{e}^{+\mathrm{i}\,\omega t} - \mathrm{e}^{-\mathrm{i}\,\omega t}\right) \\ 0 & \frac{1}{\mathrm{i}2}\left(\mathrm{e}^{+\mathrm{i}\,\omega t} - \mathrm{e}^{-\mathrm{i}\,\omega t}\right) & \frac{1}{2}\left(\mathrm{e}^{+\mathrm{i}\,\omega t} + \mathrm{e}^{-\mathrm{i}\,\omega t}\right) \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega t) & -\sin(\omega t) \\ 0 & \sin(\omega t) & \cos(\omega t) \end{pmatrix}. \end{aligned}$$

In other words, the spin rotates around the x_1-axis with frequency ω.

(ii) Seeing as

$$H \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0, \qquad [1]$$

we can immediately deduce

$$n(t) = \mathrm{e}^{tH} n^{(0)} \overset{[1]}{=} \sum_{k=0}^{\infty} \frac{t^k}{k!} H^k n^{(0)} = \frac{t^0}{0!} n^{(0)} + 0 = n^{(0)} \overset{[1]}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, n_1 is a conserved quantity.

(iii) e^{tH} is a rotation matrix, and thus the length of the vector is conserved. We can also compute this explicitly. First, we derive the square of the length:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|n(t)\|^2 \overset{[1]}{=} \dot{n}(t) \cdot n(t) + n(t) \cdot \dot{n}(t)$$

Now we plug in the definition of $\dot{n}(t)$, set $e_1 = (1, 0, 0)$ and use that the vector $a \times b$ is orthogonal to both, a and b,

$$n(t) \cdot \dot{n}(t) = n(t) \cdot \big(e_1 \times n(t)\big) \overset{[1]}{=} 0.$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}t} \|n(t)\|^2 = 0$ and the length of the vector is conserved. [1]

4. Existence of solutions within domains (4 points)

Determine for which $\lambda \in \mathbb{C}$ the second-order ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} u = \lambda u \tag{2}$$

has solutions depending on conditions on the function u:

(i) $u \in \mathcal{C}(\mathbb{R})$ where $\mathcal{C}(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$

(ii) $u \in \mathcal{C}(\mathbb{R})$ is bounded

(iii) $u \in \mathcal{C}_\infty(\mathbb{R})$ where $\mathcal{C}_\infty(\mathbb{R})$ are the continuous functions which approach 0 as $x \to \pm\infty$

(iv) $u \in \mathcal{C}([0,1])$ and $u(0) = 0 = u(1)$

Solution:

The solutions to (2) are of the form $\mathrm{e}^{\omega x}$ with $\lambda = \omega^2 \neq 0$,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \mathrm{e}^{\omega x} = \omega^2 \, \mathrm{e}^{\omega x}.$$

Clearly, the exponential function $x \mapsto \mathrm{e}^{\omega x}$ is continuous.

For $\lambda = 0$, in addition to $u(x) = 1 = \mathrm{e}^{0\,x}$, there is the solution $u(x) = x$.

(i) The equation has solutions for all $\lambda \in \mathbb{C}$. [1]

(ii) $\mathrm{e}^{\omega x}$ is bounded if and only if $\omega \in \mathrm{i}\mathbb{R}$ is purely imaginary. But then $\lambda = \mathrm{i}^2 \, |\omega|^2 = -|\omega|^2$ is always real and negative. [1]

(iii) If in addition we assume that u approaches 0 as $x \to \pm\infty$, then only the zero function $x \mapsto 0$ is a valid solution. In other words, the equation has only the trivial solution $u = 0$. [1]

(iv) Since continuous functions u on the interval $[0,1]$ with $u(0) = u(1)$ can also be thought of as periodic functions on $\mathbb{R}$, we deduce from (ii) that only $\omega = \mathrm{i}\,\pi n$, $n \in \mathbb{Z}$, are admissible ("half a wavelength" needs to fit into the interval). Thus, we obtain $\lambda = -\pi^2 \, n^2$. [1]