UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Problem Sheet 1
(2014.09.12)

Quantum Mechanics for Spin Systems & the Uncertainty Principle

Homework Problems

1. The Pauli matrices

Consider the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(i) Prove $\sigma_j \, \sigma_k = \delta_{jk} \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \, \sigma_l$ where ϵ_{jkl} is the epsilon tensor.

(ii) Prove that any 2×2 matrix can be written as the linear combination of the identity and the three Pauli matrices with coefficients h_0 and $h = (h_1, h_2, h_3)$,

$$\mathrm{Mat}_{\mathbb{C}}(2) \ni A = (a_{jk})_{1 \leq j,k \leq 2} = h_0 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^{3} h_j \, \sigma_j =: h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma. \tag{1}$$

Hint: Use that $\mathrm{Mat}_{\mathbb{C}}(2)$ is finite-dimensional.

(iii) Now assume that the coefficients $h_0, \ldots, h_3$ in equation (1) are real. Show that then the resulting matrix $H = h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma$ is hermitian. Compute the eigenvalues $E_{\pm}(h_0, h)$ of H in terms of the coefficients h_0 and h.

(iv) Use (i) to prove that for real $h_0, \ldots, h_3$

$$P_{\pm}(h_0, h) = \frac{1}{2} \left(\mathrm{id}_{\mathbb{C}^2} \pm \frac{h \cdot \sigma}{|h|} \right) , \qquad h \neq 0 \in \mathbb{R}^3, \; |h| := \sqrt{h_1^2 + h_2^2 + h_3^2},$$

are the projections onto the eigenspaces for the two eigenvalues $E_{\pm}(h_0, h)$ of H.

(v) Compute the trace of H.

Note: In physics especially, one frequently writes $h \cdot \sigma$ for $\sum_{j=1}^{3} h_j \, \sigma_j$ where $h = (h_1, h_2, h_3)$.

2. Functional calculus for 2×2 matrices

Let f be a piecewise continuous function and $H = H^*$ a hermitian 2×2 matrix. Then define

$$f(H) := \sum_{j=\pm} f(E_\pm) \, P_\pm \tag{2}$$

where $E_\pm$ are the eigenvalues of H and $P_\pm$ the two projections from problem 1.

(i) Compute $f(H)$ defined as in equation (2) for $H = h \cdot \sigma$, $h \neq 0$, and

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} .$$

(ii) Show that $f(H)$ for $f(x) = \mathrm{e}^{-\mathrm{i}tx}$ (defined via (2)) coincides with the matrix exponential, i. e.

$$f(H) = \mathrm{e}^{-\mathrm{i}t h_0} \left(\cos\bigl(|h| \, t\bigr) - \frac{\mathrm{i}}{|h|} \sin\bigl(|h| \, t\bigr) \, h \cdot \sigma \right) = \mathrm{e}^{-\mathrm{i}tH} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!} \, H^n. \tag{3}$$

Hint: Use $\mathrm{e}^{-\mathrm{i}t(h_0 + h \cdot \sigma)} = \mathrm{e}^{-\mathrm{i}t h_0} \, \mathrm{e}^{-\mathrm{i}t h \cdot \sigma}$.

(iii) Assuming h_0, h_1, h_2, h_3 are real, compute $\psi(t)$ for the initial condition $\psi(0) = \psi_0 \in \mathbb{C}^2$:

(a) $\frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = \bigl(h_2 \, \sigma_2 + h_3 \, \sigma_3\bigr) \psi(t)$

(b) $\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = h_2 \, \sigma_2 \psi(t)$

(c) $-\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = \bigl(h_0 \, \mathrm{id}_{\mathbb{C}^2} + h_3 \, \sigma_3\bigr) \psi(t)$

3. Uncertainty of Gauß functions

Compute the right-hand side of Heisenberg's uncertainty principle

$$\sigma_\psi(\hat{x}) \, \sigma_\psi(-\mathrm{i}\hbar \partial_x)$$

in one dimension for

(i) $\psi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}} \mathrm{e}^{-\frac{\lambda}{2} x^2}$, $\lambda > 0$, and

(ii) $\varphi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}} \mathrm{e}^{+\mathrm{i}x\xi_0} \, \mathrm{e}^{-\frac{\lambda}{2}(x - x_0)^2}$, $\lambda > 0$, $x_0, \xi_0 \in \mathbb{R}$.

Here, the standard deviation

$$\sigma_\psi(H) := \sqrt{\mathbb{E}_\psi\Bigl(\bigl(H - \mathbb{E}_\psi(H)\bigr)^2\Bigr)}$$

for a selfadjoint operator $H = H^*$ with respect to ψ, $\|\psi\| = 1$, is defined as in the lecture notes via the expectation value

$$\mathbb{E}_\psi(H) := \bigl\langle \psi, H \psi \bigr\rangle.$$

4. The framework of physical theories (optional)

Identify (1) states, (2) observables and (3) dynamical equations for the following physical theories:

(i) Classical mechanics on $\mathbb{R}^d$

(ii) Classical electromagnetism

Hand in home work on: Friday, 19 September 2014, before class