UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 1
(2013.09.10)

Classification of Differential Equations & Solution to the Exponential Equation

Homework Problems

1. **Classification of differential equation (7 points)**

   Classify the following differential equations: are they ODEs ore PDEs, linear homogeneous, linear-inhomogeneous or non-linear?

   (i) $\mathrm{i}\partial_t u = -\partial_x^2 u + V u$ (V is a real-valued function)

   (ii) $\partial_t \big(u^2\big) = u$

   (iii) $\partial_t u = \partial_x^2 u - V u + f(t)$ (V and $f(t)$ are real-valued functions)

   (iv) $\mathrm{i}\partial_t u = -\partial_x^2 u + \lvert u \rvert^2 \, u$

   (v) $\partial_x u = 0$ where u is a function of x and t

   (vi) $\partial_t u + u \, \partial_x u + \partial_x^3 u = 0$

   (vii) $\partial_t u + \partial_x \big(u^2\big) = 0$

2. **The matrix-valued exponential equation (15 points)**

   Let H be a $n \times n$ matrix with complex entries and define the matrix exponential

   $$\mathrm{e}^{tH} := \sum_{k=0}^{\infty} \frac{t^k}{k!} \, H^k .$$

   (We set $H^0 := \mathrm{id}_{\mathbb{C}^n}$ to the $n \times n$ identity matrix.) In (i)–(iii), show that $x(t) = \mathrm{e}^{tH} x_0$ solves

   $$\dot{x}(t) = H x(t), \qquad x(0) = x_0 \in \mathbb{C}^n. \tag{1}$$

   (i) Show that $\mathrm{e}^{t_1 H} \, \mathrm{e}^{t_2 H} = \mathrm{e}^{(t_1 + t_2) H}$ holds for all $t_1, t_2 \in \mathbb{R}$.

   (ii) Prove that $\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{tH} = H \, \mathrm{e}^{tH}$ using the definition of the derivative as a limit. (You may interchange limits and infinite sums without proof.)

   (iii) Show that $\Phi_t := \mathrm{e}^{tH}$ is the flow associated to the ODE (1).

3. Dynamics of a classical spin (14 points)

Consider the equation

$$\begin{pmatrix} \dot{n}_1 \\ \dot{n}_2 \\ \dot{n}_3 \end{pmatrix} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \qquad n(0) = n^{(0)} \in \mathbb{R}^3,$$

where $a \times b$ is the usual crossed product of vectors in $\mathbb{R}^3$ and $\omega \in \mathbb{R}$.

(i) Compute the flow.

(ii) Find the solution $n(t)$ for the particular initial condition $n^{(0)} = (1, 0, 0)$ *without* explicitly computing the matrix exponential. (Hint: Work smart, not hard.)

(iii) Show $\lVert n(t) \rVert = \lVert n^{(0)} \rVert$, i. e. verify that the length of the spin vector $\lVert n \rVert := \sqrt{n \cdot n} = \sqrt{n_1^2 + n_2^2 + n_3^2}$ is conserved.

4. Existence of solutions within domains (4 points)

Determine for which $\lambda \in \mathbb{C}$ the second-order ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} u = \lambda u \tag{2}$$

has solutions depending on conditions on the function u:

(i) $u \in \mathcal{C}(\mathbb{R})$ where $\mathcal{C}(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$

(ii) $u \in \mathcal{C}(\mathbb{R})$ is bounded

(iii) $u \in \mathcal{C}_\infty(\mathbb{R})$ where $\mathcal{C}_\infty(\mathbb{R})$ are the continuous functions which approach 0 as $x \to \pm\infty$

(iv) $u \in \mathcal{C}([0, 1])$ and $u(0) = 0 = u(1)$

Hand in home work on: Thursday, 19 September 2013, before class