UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Solutions 2
(2014.09.19)

Hilbert Spaces & Quantum Mechanical States

Homework Problems

5. Weighted L^2-spaces (16 points)

Let $\mu \in L^\infty(\mathbb{R}^d)$ be a function bounded away from 0 and $+\infty$, i. e. there exist $c, C > 0$ such that

$$0 < c \leq \mu(x) \leq C < +\infty$$

holds for almost all $x \in \mathbb{R}^d$. Define the weighted L^2-space $L^2_\mu(\mathbb{R}^d)$ as the pre-Hilbert space with scalar product

$$\langle f, g \rangle_\mu := \int_{\mathbb{R}^d} \mathrm{d}x \, \mu(x) \, \overline{f(x)} \, g(x) \tag{1}$$

so that $\|f\|_\mu := \sqrt{\langle f, f \rangle_\mu} < \infty$.

The standard (unweighted) $L^2(\mathbb{R}^d)$ space is defined as usual, i. e. we set $\mu(x) = 1$ in the above.

(i) Show that $f \in L^2(\mathbb{R}^d)$ if and only if $f \in L^2_\mu(\mathbb{R}^d)$.

(ii) Show that the map

$$U_\mu : L^2_\mu(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d) \, , \; f \mapsto \sqrt{\mu} f \, ,$$

is norm-preserving, i. e. $\|f\|_\mu = \|U_\mu f\|_{L^2(\mathbb{R}^d)}$ holds for all $f \in L^2(\mathbb{R}^d)$.

(iii) Show that $L^2_\mu(\mathbb{R}^d)$ is indeed a Hilbert space, i. e. prove that it is complete.

Solution:

(i) "⇒:" Let $f \in L^2(\mathbb{R}^d)$. Then by definition $\|f\| < \infty$, and hence also

$$\|f\|^2_\mu \overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \mu(x) \, \big|f(x)\big|^2 \overset{[1]}{\leq} \int_{\mathbb{R}^3} \mathrm{d}x \, C \, \big|f(x)\big|^2 \overset{[1]}{=} C\|f\|^2 < \infty \, .$$

"⇐:" Now assume $f \in L^2_\mu(\mathbb{R}^d)$. Since $0 < 1/\mu(x) \leq 1/c < +\infty$, we deduce

$$\begin{aligned} \|f\|^2 &\overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \big|f(x)\big|^2 = \int_{\mathbb{R}^3} \mathrm{d}x \, \frac{\mu(x)}{\mu(x)} \, \big|f(x)\big|^2 \\ &\overset{[1]}{\leq} c^{-1} \int_{\mathbb{R}^3} \mathrm{d}x \, \mu(x) \, \big|f(x)\big|^2 \overset{[1]}{=} c^{-1} \, \|f\|^2_\mu \, . \end{aligned}$$

(ii) Let $f \in L^2(\mathbb{R}^d)$. Then we compute

$$\begin{aligned}\left\|U_\mu f\right\|^2_{L^2(\mathbb{R}^d)} &\overset{[1]}{=} \langle\sqrt{\mu} f, \sqrt{\mu} f\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathrm{d}x \left|\sqrt{\mu}(x)\, f(x)\right|^2 \\ &\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}x\, \mu(x) \left|f(x)\right|^2 = \langle f, f\rangle_\mu \overset{[1]}{=} \|f\|^2_\mu \,.\end{aligned}$$

Hence, U_μ is norm-preserving.

(iii) Let $\{f_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $L^2_\mu(\mathbb{R}^d)$ [1]. That means $\left\{U_\mu f_j\right\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$ [1]. Seeing as $L^2(\mathbb{R}^d)$ is complete, $U_\mu f_j$ converges to some $g \in L^2(\mathbb{R}^d)$ [1].

Since U_μ is norm-preserving and linear, it is also invertible [1]. Moreover, the inverse $U_\mu^{-1} = U_{\mu^{-1}} : L^2(\mathbb{R}^d) \longrightarrow L^2_\mu(\mathbb{R}^d)$ is also norm-preserving by (ii) [1].

Thus, f_j converges to $U_\mu^{-1} g$ in $L^2_\mu(\mathbb{R}^d)$ [1],

$$\begin{aligned}\left\|f_j - U_\mu^{-1} g\right\|_\mu &= \left\|U_\mu^{-1}\big(U_\mu f_j - g\big)\right\|_\mu \\ &\overset{\text{(ii)}}{=} \left\|U_\mu f_j - g\right\|_{L^2(\mathbb{R}^d)} \xrightarrow{j\to\infty} 0\,.\end{aligned}$$

Hence, $L^2_\mu(\mathbb{R}^d)$ is complete, and thus a Hilbert space [1].

6. Decomposition of $L^2(\mathbb{R}^2)$ into symmetric and anti-symmetric part (20 points)

A function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ is called symmetric if $f(x, y) = f(y, x)$ and antisymmetric if $f(x, y) = -f(y, x)$ hold for all $x, y \in \mathbb{R}$.

(i) Show that $L^2_{\mathrm{s}}(\mathbb{R}^2) := \big\{ f \in L^2(\mathbb{R}^2) \mid f \text{ symmetric} \big\}$ is a closed (linear) subspace of $L^2(\mathbb{R}^2)$, i. e. prove that $L^2_{\mathrm{s}}(\mathbb{R}^2)$ is a linear subspace of the Hilbert space $L^2(\mathbb{R}^2)$, and that Cauchy sequences in $L^2_{\mathrm{s}}(\mathbb{R}^2)$ converge in $L^2_{\mathrm{s}}(\mathbb{R}^2)$.

Remark: Also $L^2_{\mathrm{as}}(\mathbb{R}^2) := \big\{ f \in L^2(\mathbb{R}^2) \mid f \text{ antisymmetric} \big\}$ is a closed subspace of $L^2(\mathbb{R}^2)$.

(ii) Show that any $f \in L^2(\mathbb{R}^2)$ can be *uniquely* decomposed $f = f_{\mathrm{s}} + f_{\mathrm{as}}$ into a symmetric part $f_{\mathrm{s}} \in L^2_{\mathrm{a}}(\mathbb{R}^2)$ and an antisymmetric part $f_{\mathrm{as}} \in L^2_{\mathrm{as}}(\mathbb{R}^2)$.

(iii) What is the physical significance of $L^2_{\mathrm{s}}(\mathbb{R}^2)$ and $L^2_{\mathrm{as}}(\mathbb{R}^2)$?

Solution:

(i) It is clear that $L^2_{\mathrm{s}}(\mathbb{R}^2)$ is a vector space with scalar product [1]. We just have to show that Cauchy sequences $\{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathrm{s}}(\mathbb{R}^2)$ converge in $L^2_{\mathrm{s}}(\mathbb{R}^2)$, i. e. that limits of symmetric functions are symmetric [1]. However, we do know that $f_n \to f$ converges in $L^2(\mathbb{R}^2)$ to some function f, because $L^2(\mathbb{R}^2)$ is a Hilbert space, and Hilbert spaces are complete [1]. Moreover, let us systematically use the notation $\widetilde{f}(x, y) := f(y, x)$.

Then we have

$$\big\| \widetilde{f} - f \big\| \overset{[1]}{\leq} \big\| \widetilde{f} - \widetilde{f} \big\| + \big\| \widetilde{f}_n - f_n \big\| + \big\| f_n - f \big\|_{L^2}.$$

We know that $\widetilde{f}_n = f_n$ since the f_n are symmetric, and we deduce that the term in the middle vanishes identically [1]. Moreover, we have

$$\begin{aligned} \big\| \widetilde{f} - \widetilde{f}_n \big\|_{L^2} &\overset{[1]}{=} \int_{\mathbb{R}^2} \mathrm{d}x \, \mathrm{d}y \, \big| \widetilde{f}(x, y) - \widetilde{f}_n(x, y) \big|^2 \\ &\overset{[1]}{=} \int_{\mathbb{R}^2} \mathrm{d}x \, \mathrm{d}y \, \big| f(y, x) - f_n(y, x) \big|^2 \\ &\overset{[1]}{=} \big\| f - f_n \big\|_{L^2}, \end{aligned}$$

and consequently

$$\big\| \widetilde{f} - f \big\| \overset{[1]}{\leq} 2 \big\| f_n - f \big\|.$$

Since the right-hand side goes to 0 as $n \to \infty$, we know that $\widetilde{f} = f$ holds, the limit function is symmetric [1].

(ii) Any function f on $\mathbb{R}^2$ can be split

$$\begin{aligned} f_{\mathrm{s}}(x, y) &\overset{[1]}{=} \tfrac{1}{2} \big(f(x, y) + f(y, x) \big) \\ f_{\mathrm{as}}(x, y) &\overset{[1]}{=} \tfrac{1}{2} \big(f(x, y) - f(y, x) \big) \end{aligned}$$

into symmetric and antisymmetric part, $f = f_{\mathrm{s}} + f_{\mathrm{as}}$ [1]. By the triangle inequality, f_{s} and f_{as} are again square-integrable, e. g.

$$\| f_{\mathrm{s}} \| \leq \tfrac{1}{2} \| f \| + \tfrac{1}{2} \| \widetilde{f} \| = \| f \| \qquad [1]$$

and analogously for f_{as}.

It remains to show that the decomposition is unique [1]. Assume $f = g_{\mathsf{s}} + g_{\mathsf{as}}$ is another decomposition of f into symmetric and antisymmetric part. Then we subtract the two decompositions from one another:

$$\big(f_{\mathsf{s}} - g_{\mathsf{s}}\big)(x,y) + \big(f_{\mathsf{as}} - g_{\mathsf{as}}\big)(x,y) \overset{[1]}{=} 0$$

After adding the above equation for (x,y) to that evaluated (y,x), and exploiting the symmetry and antisymmetry of the terms, one deduces that

$$f_{\mathsf{s}}(x,y) \overset{[1]}{=} g_{\mathsf{s}}(x,y)$$

holds for almost all $x, y \in \mathbb{R}$. A similar argument yields that $f_{\mathsf{as}}(x,y) = g_{\mathsf{as}}(x,y)$ almost everywhere [1], and hence, the decomposition is unique [1].

(iii) $L^2(\mathbb{R}^2)$ can be seen as the Hilbert space of two quantum particles moving in $\mathbb{R}$ [1]. Quantum particles are either bosons or fermions, and the two particle species can be distinguished from one another in multiparticle systems: the wave functions of bosonic particles are *symmetric*, $\psi(x,y) = \psi(y,x)$ [1] while fermionic wave functions satisfy $\psi(x,y) = -\psi(y,x)$ [1].

The above decomposition tells us that fermionic and bosonic wave functions live in different sub spaces, and if these sub spaces were not closed, then one could not distinguish between bosons and fermions in practice: if there existed a sequence of symmetric (bosonic) wave functions ψ_n which converged to an antisymmetric (fermionic) wave function ψ, then one could approximate fermionic “behavior” through bosonic wave functions.

7. Positive operators and the trace (28 points)

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$ and ρ a *density operator*, i. e. $\rho^* = \rho$, $\rho \geq 0$ and

$$\mathrm{Tr}\,\rho = \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n\rangle = 1.$$

(i) Show that the trace is independent of the choice of basis $\{\varphi_n\}_{n\in\mathbb{N}}$.

(ii) Show that any rank-1 projection $P = \langle \psi_*, \cdot\,\rangle\, \psi_*$, $\|\psi_*\| = 1$, is a density operator.

(iii) Show that $\rho^2 = \rho$ if and only if ρ is a rank-1 projection.

Remark: A bounded operator ρ on a Hilbert space $\mathcal{H}$ is selfadjoint ($\rho^* = \rho$) if and only if

$$\langle \rho\psi, \varphi\rangle = \langle \psi, \rho\varphi\rangle$$

holds for all $\varphi, \psi \in \mathcal{H}$.

Solution:

(i) By assumption, the sum

$$\mathrm{Tr}\,\rho = \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n\rangle = 1$$

converges to 1 [1], and the positivity of ρ implies it also converges *absolutely* to 1 [1].

To show that the sum is independent of the choice of orthonormal basis, let $\{\psi_j\}_{j\in\mathbb{N}}$ be a second orthonormal basis. Then we can express any φ_n from the first orthonormal basis in terms of the ψ_j,

$$\varphi_n \overset{[1]}{=} \sum_{j\in\mathbb{N}} \langle \psi_j, \varphi_n\rangle\; \psi_j\,.$$

Plugged into the sum, we obtain

$$\begin{aligned} 1 = \mathrm{Tr}\,\rho &\overset{[1]}{=} \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n\rangle \overset{[1]}{=} \sum_{j,l,n\in\mathbb{N}} \Big\langle \langle \psi_j, \varphi_n\rangle\; \psi_j, \rho\; \langle \psi_l, \varphi_n\rangle\; \psi_l \Big\rangle \\ &\overset{[1]}{=} \sum_{j,l,n\in\mathbb{N}} \overline{\langle \psi_j, \varphi_n\rangle}\; \langle \psi_l, \varphi_n\rangle\; \langle \psi_j, \rho\,\psi_l\rangle \overset{[1]}{=} \sum_{j,l,n\in\mathbb{N}} \Big\langle \psi_l, \langle \varphi_n, \psi_j\rangle\; \varphi_n \Big\rangle\; \langle \psi_j, \rho\,\psi_l\rangle \\ &\overset{[1]}{=} \sum_{j,l\in\mathbb{N}} \langle \psi_l, \psi_j\rangle\; \langle \psi_j, \rho\,\psi_l\rangle \overset{[1]}{=} \sum_{j\in\mathbb{N}} \langle \psi_j, \rho\,\psi_j\rangle\;. \end{aligned}$$

(ii) First of all, $P = \langle \psi_*, \cdot\,\rangle\; \psi_*$ is selfadjoint, because for all $\varphi, \phi \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} \big\langle \varphi, P\phi\big\rangle &\overset{[1]}{=} \Big\langle \varphi, \langle \psi_*, \phi\rangle\; \psi_* \Big\rangle = \langle \psi_*, \phi\rangle\; \langle \varphi, \psi_*\rangle \\ &\overset{[1]}{=} \Big\langle \langle \psi_*, \varphi\rangle\, \psi_*, \phi \Big\rangle \overset{[1]}{=} \big\langle P\varphi, \phi\big\rangle\,. \end{aligned}$$

Moreover, $P \geq 0$ because $P^2 = P$, and thus

$$\langle \varphi, P\varphi\rangle \overset{[1]}{=} \langle \varphi, P^2\varphi\rangle \overset{[1]}{=} \langle P^*\varphi, P\varphi\rangle = \langle P\varphi, P\varphi\rangle \overset{[1]}{\geq} 0\,.$$

By (i), we can compute the trace in any orthonormal basis, so for instance we can pick $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\varphi_1 = \psi_*$, and in that basis only one term of the sum survives,

$$\begin{aligned}\mathrm{Tr}\, P &\overset{[1]}{=} \sum_{n=1}^{\infty} \langle \varphi_n, P\varphi_n \rangle \overset{[1]}{=} \langle \psi_*, P\psi_* \rangle + \sum_{n=2}^{\infty} \langle \varphi_n, P\varphi_n \rangle \\ &\overset{[1]}{=} \langle \psi_*, \psi_* \rangle + 0 \overset{[1]}{=} 1\,.\end{aligned}$$

Thus, P is a density operator.

(iii) "$\Leftarrow$:" If ρ is a rank-1 projection, then $\rho^2 = \rho$ is a density operator by (ii) [1].

"$\Rightarrow$:" Assume $\rho^2 = \rho$, i. e. ρ is an orthogonal projection (selfadjointness is included in the definition of ρ) [1]. Hence, we can spit $L^2(\mathbb{R}^d) = \mathrm{ran}\,\rho \oplus \big(\mathrm{ran}\,\rho\big)^{\perp}$ into the range of ρ and its orthogonal complement, and the action of ρ and $\psi = \psi_\rho + \psi_\rho^{\perp}$ is

$$\rho\psi = \rho\big(\psi_\rho + \psi_\rho^{\perp}\big) = \psi_\rho\,. \qquad [1]$$

Thus, choosing a basis $\{\varphi_n\}_{n\in\mathbb{N}} = \{\varphi_n\}_{n\in\mathcal{I}} \cup \{\varphi_n\}_{n\in\mathbb{N}\setminus\mathcal{I}}$ where $\{\varphi_n\}_{n\in\mathcal{I}}$ is an orthonormal basis of $\mathrm{ran}\,\rho$, we compute

$$\begin{aligned}\mathrm{Tr}\, \rho &\overset{[1]}{=} \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n \rangle \overset{[1]}{=} \sum_{n\in\mathcal{I}} \langle \varphi_n, \rho\varphi_n \rangle \\ &\overset{[1]}{=} \sum_{n\in\mathcal{I}} \langle \varphi_n, \varphi_n \rangle \overset{[1]}{=} |\mathcal{I}| \overset{!}{=} 1\,.\end{aligned}$$

Since $|\mathcal{I}|$ is the dimensionality of $\mathrm{ran}\,\rho$ [1], we deduce that $\dim\big(\mathrm{ran}\,\rho\big) = 1$, and thus, P is a rank-1 projection [1].