Classification of Differential Equations & Solution to the Exponential Equation

Homework Problems

5. Uniqueness of solutions of ordinary differential equations (13 points)

Consider the ODE

$$\dot{x} = -|x|^{\alpha}, \qquad x(0) = 1. \tag{1}$$

(i) Find solutions to this ODE for

(a) $\alpha = 2$ and

(b) $\alpha = 1$,

and give the longest time interval on which these solutions exist.

(ii) In case of $\alpha = 1/2$, the ODE does *not* have a unique solution: Show that for any $t_0 > 2$

$$x(t) = \begin{cases} \frac{1}{4}(t-2)^2 & t \leq 2 \\ 0 & 2 < t \leq t_0 \\ -\frac{1}{4}(t-t_0)^2 & t > t_0 \end{cases}$$

solves (1).

(iii) For two of the three values of α, the solution of (1) is either not unique or does not exist for all t. Explain in each case why that does *not* contradict the Picard-Lindelöf theorem.

Solution:

(i) (a) The differential equation is separable, so one may find solutions via the ansatz

$$t = \int_0^t \mathrm{d}s \overset{[1]}{=} -\int_{x(0)}^{x(t)} \mathrm{d}x\, x^{-2} = \left[x^{-1}\right]_1^{x(t)}$$
$$\overset{[1]}{=} \frac{1}{x(t)} - 1,$$

where we have used the initial condition $x(0) = 1$ and the fact that at least for small times, $x(t) > 0$. This equation can be inverted,

$$x(t) \overset{[1]}{=} \frac{1}{1+t}.$$

Clearly, the maximal time interval on which the solution exists is $(-1, \infty)$ [1].

(b) Again, as before, we can find the solution by integrating,

$$t = \int_0^t \mathrm{d}s \overset{[1]}{=} -\int_{x(0)}^{x(t)} \mathrm{d}x\, x^{-1} = -\bigl[\ln x\bigr]_1^{x(t)}$$
$$= -\ln x(t) + \ln 1 \overset{[1]}{=} -\ln x(t).$$

This equation can be solved explicitly for all $t \in \mathbb{R}$,

$$x(t) \overset{[1]}{=} \mathrm{e}^{-t},$$

and in fact, $x(t) > 0$ for all $t \in \mathbb{R}$ [1].

(ii) Independently of the value of t_0, we have $x(0) = \frac{1}{4}(-2)^2 = 1$, so the initial condition is always satisfied. To verify whether $x(t)$ solves the ODE, we compare

$$\dot{x}(t) \overset{[1]}{=} \begin{cases} \frac{1}{2}(t-2) & t \leq 2 \\ 0 & 2 < t \leq t_0 \\ -\frac{1}{2}(t-t_0) & t > t_0 \end{cases}$$

to

$$F\bigl(x(t)\bigr) = -\sqrt{|x(t)|} \overset{[1]}{=} \begin{cases} -\frac{1}{2}\,|t-2| & t \leq 2 \\ 0 & 2 < t \leq t_0 \\ -\frac{1}{2}(t-t_0) & t > t_0 \end{cases}$$
$$\overset{[1]}{=} \begin{cases} \frac{1}{2}(t-2) & t \leq 2 \\ 0 & 2 < t \leq t_0 \\ -\frac{1}{2}(t-t_0) & t > t_0 \end{cases},$$

and see that the two agree. Hence, $x(t)$ is a solution for any $t_0 > 2$.

(iii) $\alpha = 2$: Here, the solution does not exist for all times. This is not a contradiction to the Picard-Lindelöf theorem, because the vector field $F(x) = -x^2$ is *not* Lipschitz, it grows faster than $|x|$. Hence, the assumptions of the Picard-Lindelöf theorem are not satisfied. [1]

$\alpha = 1/2$: The solution is not unique: the vector field $F(x) = -\sqrt{x}$ is only Lipschitz *away* from $x = 0$, and as soon as the trajectory reaches $x = 0$, the solution cannot be uniquely extended. Also here, the assumptions of the Picard-Lindelöf theorem are not verified. [1]

6. Using the Grönwall lemma to estimate the distance between trajectories (8 points)

Assume the vector field $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is globally Lipschitz with constant $L > 0$, and let Φ be the flow associated to $\dot{x} = F$. Moreover, let $x_0, x_0' \in \mathbb{R}^n$ be two points which are ε-close, $\lvert x_0 - x_0' \rvert = \varepsilon$.

Use the Grönwall lemma to estimate the distance between $x(t) := \Phi_t(x_0)$ and $x'(t) := \Phi_t(x_0')$ from above (similar to equations (2.2.5) and equations (2.2.6)). Make your arguments rigorously.

Solution:

Since the vector field F is globally Lipschitz, Corollary 2.2.8 applies and we know that the flow Φ exists for all $t \in \mathbb{R}$ [1]. Choose

$$u(t) := \big\lvert x(t) - x'(t) \big\rvert. \qquad [1]$$

Then $u(0) = \big\lvert x(0) - x'(0) \big\rvert = \big\lvert x_0 - x_0' \big\rvert = \varepsilon$ [1].

Repeating the arguments of equation (2.2.7), we know that

$$\begin{aligned} \left\lvert \frac{\mathrm{d}}{\mathrm{d}t} \big(x(t) - x'(t)\big) \right\rvert &= \left\lvert \lim_{\delta \to 0} \frac{\big(x(t+\delta) - x'(t+\delta)\big) - \big(x(t) - x'(t)\big)}{\delta} \right\rvert \\ &= \lim_{\delta \to 0} \frac{\big\lvert \big(x(t+\delta) - x'(t+\delta)\big) - \big(x(t) - x'(t)\big) \big\rvert}{\lvert \delta \rvert} \\ &\geq \lim_{\delta \to 0} \frac{\big\lvert x(t+\delta) - x'(t+\delta) \big\rvert - \big\lvert x(t) - x'(t) \big\rvert}{\lvert \delta \rvert} = \frac{\mathrm{d}}{\mathrm{d}t} u(t). \qquad [1] \end{aligned}$$

Thus, we can estimate the time derivative of u by u itself:

$$\begin{aligned} \dot{u}(t) \leq \big\lvert \dot{x}(t) - \dot{x}'(t) \big\rvert &\overset{[1]}{=} \big\lvert F\big(x(t)\big) - F\big(x'(t)\big) \big\rvert \\ &\overset{[1]}{=} L \, \big\lvert x(t) - x'(t) \big\rvert = L \, u(t) \end{aligned}$$

Hence, the Grönwall lemma 2.2.6 applies [1] and we obtain

$$u(t) \leq u(0) \, \mathrm{e}^{tL} \overset{[1]}{=} \varepsilon \, \mathrm{e}^{tL}.$$

7. The classical harmonic oscillator (25 points)

Consider the driven harmonic oscillator

$$\ddot{q}(t) + \omega^2 \, q(t) = f(t) \tag{2}$$

of frequency $\omega > 0$.

(i) Solve the homogeneous equation (i. e. set $f = 0$ in (2)) by rewriting it as a first-order problem (cf. Section 2.1 of the lecture notes). Determine the dimensionality of the space of solutions.

(ii) Find a system of *real-valued* solutions for the homogeneous equation.

(iii) Solve the inhomogeneous problem for the functions f and initial conditions listed below, and characterize the behavior of the solutions as $t \to \pm\infty$. Also verify that the solution satisfies the initial conditions.

(a) $f(t) = \omega^2 \in \mathbb{R}$ (constant function), $q(0) = 0$, $\dot{q}(0) = 0$

(b) $f(t) = \omega^2 \cos(\omega t)$, $q(0) = 0$, $\dot{q}(0) = \omega$

Solution:

(i) We set $y_1 := q$ and $y_2 := \dot{q}$ [1] so that we can rewrite (2) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} y_2 \\ -\omega^2 \, y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{=:H} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} .$$

The flow associated to the homogeneous equation is e^{tH}. A quick calculation reveals that H is diagonalizable, has eigenvalues $\pm\mathrm{i}\omega$ and

$$\begin{aligned} H &= U \, \mathrm{diag}(+\mathrm{i}\omega, -\mathrm{i}\omega) \, U^{-1} \\ &\overset{[3]}{=} \begin{pmatrix} 1 & 1 \\ \mathrm{i}\omega & -\mathrm{i}\omega \end{pmatrix} \begin{pmatrix} \mathrm{i}\omega & 0 \\ 0 & -\mathrm{i}\omega \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -\mathrm{i}/\omega \\ 1 & +\mathrm{i}/\omega \end{pmatrix} . \end{aligned}$$

Hence, we can compute the exponential:

$$\begin{aligned} \mathrm{e}^{tH} &\overset{[1]}{=} U \, \mathrm{e}^{t \, \mathrm{diag}(\mathrm{i}\omega, -\mathrm{i}\omega)} \, U^{-1} \\ &\overset{[1]}{=} \begin{pmatrix} 1 & 1 \\ \mathrm{i}\omega & -\mathrm{i}\omega \end{pmatrix} \begin{pmatrix} \mathrm{e}^{+it\omega} & 0 \\ 0 & \mathrm{e}^{-it\omega} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -\mathrm{i}/\omega \\ 1 & +\mathrm{i}/\omega \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \mathrm{e}^{+it\omega} & \mathrm{e}^{-it\omega} \\ \mathrm{i}\omega \mathrm{e}^{+it\omega} & -\mathrm{i}\omega \mathrm{e}^{-it\omega} \end{pmatrix} \begin{pmatrix} 1 & -\mathrm{i}/\omega \\ 1 & +\mathrm{i}/\omega \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \mathrm{e}^{+it\omega} + \mathrm{e}^{-it\omega} & \frac{\mathrm{i}}{\omega} \big(-\mathrm{e}^{+it\omega} + \mathrm{e}^{-it\omega} \big) \\ \mathrm{i}\omega \big(\mathrm{e}^{+it\omega} - \mathrm{e}^{-it\omega} \big) & \mathrm{e}^{+it\omega} + \mathrm{e}^{-it\omega} \end{pmatrix} \\ &\overset{[1]}{=} \begin{pmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{pmatrix} \end{aligned}$$

(Note that the lower row is the time derivative of the upper row – as it should be.)

The dimension of the system of solutions is 2 [1].

(ii) A set of real solutions to (2) would be $\cos \omega t$ [1] and $\sin \omega t$ [1].

(iii) The general solution to the inhomogeneous problem here is

$$\begin{aligned} y(t) &= \mathrm{e}^{tH}\begin{pmatrix} q(0) \\ \dot{q}(0) \end{pmatrix} + \int_0^t \mathrm{d}s\, \mathrm{e}^{(t-s)H}\begin{pmatrix} 0 \\ f(t) \end{pmatrix} \\ &= \begin{pmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} q(0) \\ \dot{q}(0) \end{pmatrix} + \int_0^t \mathrm{d}s \begin{pmatrix} \frac{1}{\omega} f(s)\sin\omega(t-s) \\ f(s)\cos\omega(t-s) \end{pmatrix} \end{aligned}$$

(a) Here, $y_0 = \big(q(0), \dot{q}(0)\big) = 0$ [1].

$$\begin{aligned} y(t) &\overset{[1]}{=} \begin{pmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^t \mathrm{d}s \begin{pmatrix} \cos\omega(t-s) & \frac{1}{\omega}\sin\omega(t-s) \\ -\omega\sin\omega(t-s) & \cos\omega(t-s) \end{pmatrix}\begin{pmatrix} 0 \\ \omega^2 \end{pmatrix} \\ &\overset{[1]}{=} \int_0^t \mathrm{d}s \begin{pmatrix} \omega\sin\omega(t-s) \\ \omega^2\cos\omega(t-s) \end{pmatrix} = \left[\begin{pmatrix} +\cos\omega(t-s) \\ -\omega\sin\omega(t-s) \end{pmatrix}\right]_0^t = \begin{pmatrix} \cos 0 - \cos\omega t \\ \omega\,(0 + \sin\omega t) \end{pmatrix} \\ &\overset{[1]}{=} \begin{pmatrix} 1 - \cos\omega t \\ \omega\sin\omega t \end{pmatrix} \end{aligned}$$

The solution $q(t) = y_1(t) = \sin\omega t$ to (2) is just the first component of y. Note that $y(0) = (0,0) = \big(q(0), \dot{q}(0)\big)$ [1]. The solution $y(t)$ oscillates, and hence it remains bounded as $t \to \pm\infty$ [1].

(b) To solve this for $y_0 = (0, \omega)$ [1], we just need to add the homogeneous solution for y_0 to the term containing the inhomogeneity. In order to compute the latter explicitly, we need

$$[1] \qquad \begin{cases} \sin\omega(s-t)\,\cos\omega s = \frac{1}{2}\sin\omega(2s-t) - \frac{1}{2}\sin\omega t \\ \cos\omega(s-t)\,\cos\omega s = \frac{1}{2}\cos\omega(2s-t) + \frac{1}{2}\cos\omega t \end{cases}.$$

(These equations can be derived by writing sin and cos in terms of complex exponentials.) Hence, we obtain

$$\begin{aligned} \int_0^t \mathrm{d}s \begin{pmatrix} \frac{1}{\omega} f(s)\sin\omega(t-s) \\ f(s)\cos\omega(t-s) \end{pmatrix} &\overset{[1]}{=} \frac{1}{2}\int_0^t \mathrm{d}s \begin{pmatrix} \omega\sin\omega(2s-t) - \omega\sin\omega t \\ \omega^2\cos\omega(2s-t) + \omega^2\cos\omega t \end{pmatrix} \\ &= \frac{1}{2}\left[\begin{pmatrix} -\frac{1}{2}\cos\omega(2s-t) - \omega\, s\sin\omega t \\ \frac{\omega}{2}\sin\omega(2s-t) + \omega^2\, s\cos\omega t \end{pmatrix}\right]_0^t \\ &= \frac{1}{4}\begin{pmatrix} -\cos\omega t - 2\omega\, t\sin\omega t + \cos\omega t \\ \omega\sin\omega t + 2\omega^2\, t\cos\omega t + \omega\sin\omega t \end{pmatrix} \\ &\overset{[1]}{=} \frac{1}{2}\begin{pmatrix} -\omega\, t\sin\omega t \\ \omega\sin\omega t + \omega^2\, t\cos\omega t \end{pmatrix} \end{aligned}$$

Now the total solution is just the sum,

$$\begin{aligned} y(t) &\overset{[1]}{=} \begin{pmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} 0 \\ \omega \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\omega\, t\sin\omega t \\ \omega\sin\omega t + \omega^2\, t\cos\omega t \end{pmatrix} \\ &\overset{[1]}{=} \begin{pmatrix} \sin\omega t \\ \omega\cos\omega t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\omega\, t\sin\omega t \\ \omega\sin\omega t + \omega^2\, t\cos\omega t \end{pmatrix} \end{aligned}$$

Again, $y(0) = (0, \omega)$ satisfies the initial conditions [1]. $\lim_{t\to\pm\infty}\big|y_j(t)\big| = \infty$ for $j = 1, 2$ [1].