UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Problem Sheet 2
(2014.09.19)

Hilbert Spaces & Quantum Mechanical States

Homework Problems

5. Weighted L^2-spaces

Let $\mu \in L^\infty(\mathbb{R}^d)$ be a function bounded away from 0 and $+\infty$, i. e. there exist $c, C > 0$ such that

$$0 < c \leq \mu(x) \leq C < +\infty$$

holds for almost all $x \in \mathbb{R}^d$. Define the weighted L^2-space $L^2_\mu(\mathbb{R}^d)$ as the pre-Hilbert space with scalar product

$$\langle f, g \rangle_\mu := \int_{\mathbb{R}^d} \mathrm{d}x \, \mu(x) \, \overline{f(x)} \, g(x) \tag{1}$$

so that $\lVert f \rVert_\mu := \sqrt{\langle f, f \rangle_\mu} < \infty$.

The standard (unweighted) $L^2(\mathbb{R}^d)$ space is defined as usual, i. e. we set $\mu(x) = 1$ in the above.

(i) Show that $f \in L^2(\mathbb{R}^d)$ if and only if $f \in L^2_\mu(\mathbb{R}^d)$.

(ii) Show that the map

$$U_\mu : L^2_\mu(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d) \, , \; f \mapsto \sqrt{\mu} f \, ,$$

is norm-preserving, i. e. $\lVert f \rVert_\mu = \lVert U_\mu f \rVert_{L^2(\mathbb{R}^d)}$ holds for all $f \in L^2(\mathbb{R}^d)$.

(iii) Show that $L^2_\mu(\mathbb{R}^d)$ is indeed a Hilbert space, i. e. prove that it is complete.

6. Decomposition of $L^2(\mathbb{R}^2)$ into symmetric and anti-symmetric part

A function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ is called symmetric if $f(x,y) = f(y,x)$ and antisymmetric if $f(x,y) = -f(y,x)$ hold for all $x, y \in \mathbb{R}$.

(i) Show that $L^2_{\mathrm{s}}(\mathbb{R}^2) := \big\{ f \in L^2(\mathbb{R}^2) \mid f \text{ symmetric} \big\}$ is a closed (linear) subspace of $L^2(\mathbb{R}^2)$, i. e. prove that $L^2_{\mathrm{s}}(\mathbb{R}^2)$ is a linear subspace of the Hilbert space $L^2(\mathbb{R}^2)$, and that Cauchy sequences in $L^2_{\mathrm{s}}(\mathbb{R}^2)$ converge in $L^2_{\mathrm{s}}(\mathbb{R}^2)$.

Remark: Also $L^2_{\mathrm{as}}(\mathbb{R}^2) := \big\{ f \in L^2(\mathbb{R}^2) \mid f \text{ antisymmetric} \big\}$ is a closed subspace of $L^2(\mathbb{R}^2)$.

(ii) Show that any $f \in L^2(\mathbb{R}^2)$ can be *uniquely* decomposed $f = f_{\mathrm{s}} + f_{\mathrm{as}}$ into a symmetric part $f_{\mathrm{s}} \in L^2_{\mathrm{a}}(\mathbb{R}^2)$ and an antisymmetric part $f_{\mathrm{as}} \in L^2_{\mathrm{as}}(\mathbb{R}^2)$.

(iii) What is the physical significance of $L^2_{\mathrm{s}}(\mathbb{R}^2)$ and $L^2_{\mathrm{as}}(\mathbb{R}^2)$?

7. Positive operators and the trace

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$ and ρ a *density operator*, i. e. $\rho^* = \rho$, $\rho \geq 0$ and

$$\mathrm{Tr}\,\rho = \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n \rangle = 1.$$

(i) Show that the trace is independent of the choice of basis $\{\varphi_n\}_{n\in\mathbb{N}}$.

(ii) Show that any rank-1 projection $P = \langle \psi_*, \cdot \rangle \, \psi_*$, $\|\psi_*\| = 1$, is a density operator.

(iii) Show that $\rho^2 = \rho$ if and only if ρ is a rank-1 projection.

Remark: A bounded operator ρ on a Hilbert space $\mathcal{H}$ is selfadjoint ($\rho^* = \rho$) if and only if

$$\langle \rho\psi, \varphi \rangle = \langle \psi, \rho\varphi \rangle$$

holds for all $\varphi, \psi \in \mathcal{H}$.

Hand in home work on: Friday, 26 September 2014, before class