Classification of Differential Equations & Solution to the Exponential Equation

Homework Problems

5. Uniqueness of solutions of ordinary differential equations (13 points)

Consider the ODE

$$\dot{x} = - |x|^{\alpha} , \qquad\qquad x(0) = 1. \tag{1}$$

(i) Find solutions to this ODE for

(a) $\alpha = 2$ and

(b) $\alpha = 1$,

and give the longest time interval on which these solutions exist.

(ii) In case of $\alpha = 1/2$, the ODE does *not* have a unique solution: Show that for any $t_0 > 2$

$$x(t) = \begin{cases} \frac{1}{4}(t-2)^2 & t \leq 2 \\ 0 & 2 < t \leq t_0 \\ -\frac{1}{4}(t-t_0)^2 & t > t_0 \end{cases}$$

solves (1).

(iii) For two of the three values of α, the solution of (1) is either not unique or does not exist for all t. Explain in each case why that does *not* contradict the Picard-Lindelöf theorem.

6. Using the Grönwall lemma to estimate the distance between trajectories (8 points)

Assume the vector field $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is globally Lipschitz with constant $L > 0$, and let Φ be the flow associated to $\dot{x} = F$. Moreover, let $x_0, x_0' \in \mathbb{R}^n$ be two points which are ε-close, $\left|x_0 - x_0'\right| = \varepsilon$.

Use the Grönwall lemma to estimate the distance between $x(t) := \Phi_t(x_0)$ and $x'(t) := \Phi_t(x_0')$ from above (similar to equations (2.2.5) and equations (2.2.6)). Make your arguments rigorously.

7. The classical harmonic oscillator (25 points)

Consider the driven harmonic oscillator

$$\ddot{q}(t) + \omega^2 \, q(t) = f(t) \tag{2}$$

of frequency $\omega > 0$.

(i) Solve the homogeneous equation (i. e. set $f = 0$ in (2)) by rewriting it as a first-order problem (cf. Section 2.1 of the lecture notes). Determine the dimensionality of the space of solutions.

(ii) Find a system of *real-valued* solutions for the homogeneous equation.

(iii) Solve the inhomogeneous problem for the functions f and initial conditions listed below, and characterize the behavior of the solutions as $t \to \pm\infty$. Also verify that the solution satisfies the initial conditions.

   (a) $f(t) = \omega^2 \in \mathbb{R}$ (constant function), $q(0) = 0$, $\dot{q}(0) = 0$

   (b) $f(t) = \omega^2 \cos(\omega t)$, $q(0) = 0$, $\dot{q}(0) = \omega$

Hand in home work on: Thursday, 26 September 2013, before class