UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Solutions 3
(2014.09.26)

Hilbert Spaces & Operators

Homework Problems

8. Orthogonal subspaces and projections onto subspaces (16 points)

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis (ONB) of a Hilbert space $\mathcal{H}$ and $N \in \mathbb{N}$.

(i) Prove that $E := \{\varphi_1, \ldots, \varphi_N\}^\perp$ is a sub *vector* space.

(ii) Give an ONB for the subspace $E = \{\varphi_1, \ldots, \varphi_N\}^\perp$.

(iii) Show that $\big(\{\varphi_1, \ldots, \varphi_N\}^\perp\big)^\perp = E^\perp = \mathrm{span}\{\varphi_1, \ldots, \varphi_N\}$.

Moreover, define the map

$$P : \mathcal{H} \longrightarrow \mathcal{H} \, , \; P\psi := \sum_{n=1}^{N} \langle \varphi_n, \psi \rangle \; \varphi_n \, .$$

(iv) Show that P is linear, i. e. for any $\varphi, \psi \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, we have $P\big(\alpha\varphi + \psi\big) = \alpha \, P\varphi + P\psi$.

(v) Show that P is a projection, i. e. $P^2 = P$.

(vi) Show that P is bounded, i. e. $\|P\varphi\| \leq \|\varphi\|$ holds for any $\varphi \in \mathcal{H}$.

Solution:

(i) The orthogonal complement is defined as

$$E \overset{[1]}{=} \big\{ \psi \in \mathcal{H} \mid \langle \varphi_j, \psi \rangle = 0, \; j = 1, \ldots, N \big\} \, .$$

For any $\phi, \psi \in E$ and $\alpha \in \mathbb{C}$, also the vector $\alpha \, \phi + \psi$ is an element of E [1]: for all $j = 1, \ldots, N$

$$\langle \varphi_j, \alpha \, \phi + \psi \rangle = \alpha \, \langle \varphi_j, \phi \rangle + \langle \varphi_j, \psi \rangle \overset{[1]}{=} 0$$

is satisfied. Hence, E is a linear subspace of $\mathcal{H}$.

(ii) $\{\varphi_j\}_{j=N+1}^{\infty}$ [1]

(iii) Then $\varphi_j \in E^\perp$, because by definition of E

$$\langle \varphi_j, \psi \rangle \overset{[1]}{=} 0$$

holds for all $\psi \in \mathcal{H}$. Thus, $\varphi_j \in E^\perp$ for all $j = 1, \ldots, N$. By (i), E is a linear sub space [1].

Now assume that there exists a $\psi \in E^\perp$ which is *not* a linear combination of $\{\varphi_1, \ldots, \varphi_N\}$ [1]. Since $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, we can express ψ as

$$\psi = \sum_{j=1}^{\infty} c_j \, \varphi_j \, . \qquad [1]$$

By assumption, there exists a $n \geq N+1$ for which $c_n \neq 0$ [1]. But then

$$\langle \varphi_n, \psi \rangle = c_n \neq 0$$

and ψ cannot be an element of $E^\perp$, contradiction [1].

Hence, $E^\perp = \text{span}\{\varphi_1, \ldots, \varphi_N\}$.

(iv) The linearity of P follows from the linearity of the scalar product in the first argument: for all $\phi, \psi \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} P\big(\alpha\,\phi + \psi\big) &\overset{[1]}{=} \sum_{j=1}^{N} \langle \varphi_j, \alpha\,\phi + \psi \rangle \;\varphi_j = \alpha \sum_{j=1}^{N} \langle \varphi_j, \phi \rangle \;\varphi_j + \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \;\varphi_j \\ &\overset{[1]}{=} \alpha\, P\phi + P\psi \, . \end{aligned}$$

Hence, P is linear.

(v) For any $\psi \in \mathcal{H}$, we deduce using the linearity of P:

$$\begin{aligned} P^2\psi &= P\left(\sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \;\varphi_j\right) \overset{[1]}{=} \sum_{k=1}^{N} \langle \varphi_j, \psi \rangle \; P\varphi_j \\ &= \sum_{k,j=1}^{N} \langle \varphi_j, \psi \rangle \; \underbrace{\langle \varphi_k, \varphi_j \rangle}_{=\delta_{k,j}} \varphi_k = \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \;\varphi_j \overset{[1]}{=} P\psi \end{aligned}$$

Hence, P is a projection.

(vi) With the help of Bessel's inequality [1], we obtain the claim:

$$\big\|P\psi\big\| = \left\| \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \;\varphi_j \right\| \overset{[1]}{\leq} \|\psi\|$$

9. The Fock space (13 points)

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ the n-fold tensor product. By definition we set $\mathcal{H}^{\otimes 0} := \mathbb{C}$. Define the *Fock space* over $\mathcal{H}$ as $\mathfrak{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$ with scalar product

$$\langle \varphi, \psi \rangle_{\mathfrak{F}} := \sum_{n=0}^{\infty} \langle \varphi_n, \psi_n \rangle_{\mathcal{H}^{\otimes n}}, \qquad \varphi = (\varphi_0, \varphi_1, \ldots), \psi = (\psi_0, \psi_1, \ldots) \in \mathfrak{F}(\mathcal{H}).$$

In the physics literature, $\mathcal{H}^{\otimes n}$ is called the *n-particle Fock sector*.

(i) Give a countable orthonormal basis of $\mathfrak{F}(\mathcal{H})$. (No proof is necessary.)

(ii) Show that for any element $\psi = (\psi_0, \psi_1, \psi_2, \ldots)$ of $\mathfrak{F}(\mathcal{H})$

$$\lim_{n \to \infty} \left\| \psi_n \right\|_{\mathcal{H}^{\otimes n}} = 0$$

holds where ψ_n is the corresponding element of the n-particle Fock sector.

Let A be a linear, bounded operator on $\mathcal{H}$ with domain $\mathcal{D}(A) = \mathcal{H}$. Then we define the *second quantization of* A as the operator

$$\mathrm{d}\Gamma(A) := \bigoplus_{n=0}^{\infty} \big(A \otimes \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} + \ldots + \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} \otimes A \big)$$

acting on $\mathfrak{F}(\mathcal{H})$.

(iii) Give the action of $\mathrm{d}\Gamma(A)$ on the nth Fock sector $\mathcal{H}^{\otimes n}$, i. e. find $\big(\mathrm{d}\Gamma(A)\psi\big)_n$.

(iv) Give the domain of $\mathrm{d}\Gamma(\mathrm{id}_{\mathcal{H}})$ and discuss the physical meaning of $\mathrm{d}\Gamma(\mathrm{id}_{\mathcal{H}})$.

Solution:

(i) Since $\mathcal{H}$ is countable, there exists a countable orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}}$ [1]. Then

$$\Big\{ \big(1, \varphi_{k_{1\,1}}, \varphi_{k_{2\,1}} \otimes \varphi_{k_{2\,2}}, \ldots, \varphi_{k_{n\,1}} \otimes \cdots \otimes \varphi_{k_{n\,n}}, \ldots\big) \Big\}_{\substack{n \in \mathbb{N} \\ 1 \leq j \leq n \\ k_{n\,j} \in \mathbb{N}}} \qquad [2]$$

is a countable orthonormal basis of $\mathfrak{F}(\mathcal{H})$.

(ii) Pick an arbitrary $\psi = (\psi_0, \psi_1, \ldots) \in \mathfrak{F}(\mathcal{H})$. Then

$$\|\psi\|^2 = \sum_{n=0}^{\infty} \left\| \psi_n \right\|^2_{\mathcal{H}^{\otimes n}} < \infty \qquad [1]$$

which necessarily means $\left\| \psi_n \right\|_{\mathcal{H}^{\otimes n}} \to 0$ as $n \to \infty$ [1].

(iii) $\big(\mathrm{d}\Gamma(A)\psi\big)_n = \big(A \otimes \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} + \ldots + \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} \otimes A\big)\psi_n$ [1]

(iv) The second quantization of the identity is the so-called (particle) number operator [1] which – as the name suggests – measures the number of particles. The operator acts on the nth Fock sector $\mathcal{H}^{\otimes n}$ as

$$\underbrace{\mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} + \ldots + \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}}}_{n \text{ terms}} \overset{[1]}{=} n \, \mathrm{id}_{\mathcal{H}^{\otimes n}}$$

and hence, we have

$$\mathrm{d}\Gamma(\mathrm{id}_{\mathcal{H}}) \overset{[2]}{=} \bigoplus_{n=0}^{\infty} n\, \mathrm{id}_{\mathcal{H}^{\otimes n}}.$$

The domain consists of states with a finite number of particles [1], i. e.

$$\begin{aligned}\mathcal{D}\big(\mathrm{d}\Gamma(\mathrm{id}_{\mathcal{H}})\big) &\overset{[1]}{=} \Big\{\varphi \in \mathfrak{F}(\mathcal{H}) \;\big|\; \mathrm{d}\Gamma(\mathrm{id}_{\mathcal{H}})\varphi \in \mathfrak{F}(\mathcal{H})\Big\} \\ &\overset{[1]}{=} \Big\{\varphi = (\varphi_0, \varphi_1, \ldots) \in \mathfrak{F}(\mathcal{H}) \;\big|\; \textstyle\sum_{n=0}^{\infty} n^2 \big|\big\langle \varphi_n, \varphi_n \big\rangle_{\mathcal{H}^{\otimes n}}\big|^2 < \infty\Big\}.\end{aligned}$$

10. Best approximation: Fourier series (12 points)

Let $L^2([0,2\pi])$ be the Hilbert space of square-integrable functions with scalar product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, \overline{f(x)}\, g(x).$$

(i) Show that $\{\mathrm{e}^{+\mathrm{i}nx}\}_{n\in\mathbb{Z}}$ is an orthonormal system of vectors in $L^2([0,2\pi])$.

(ii) Let $E := \{\mathrm{e}^{+\mathrm{i}nx}\}_{\substack{n\in\mathbb{Z}\\ |n|\leq 4}}$, and consider the functions $f(x) = \sin(2x)$ and $g(x) = x$. Give the element of best approximation of f and g in span E.

(iii) Why don't these arguments work for $L^2(\mathbb{R})$?

Solution:

(i) The scalar product for $n \neq m$ is

$$\begin{aligned}\left\langle \mathrm{e}^{+\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}mx}\right\rangle &\overset{[1]}{=} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}nx}\, \mathrm{e}^{+\mathrm{i}mx} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{+\mathrm{i}(m-n)x} = \frac{1}{2\pi}\frac{1}{i(m-n)}\, \mathrm{e}^{+\mathrm{i}(m-n)x}\Big|_0^{2\pi} \overset{[1]}{=} 0.\end{aligned}$$

For $n = m$ we obtain instead

$$\left\langle \mathrm{e}^{+\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}mx}\right\rangle \overset{[1]}{=} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{+\mathrm{i}(m-n)x} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, 1 \overset{[1]}{=} 1.$$

The vectors are all mutually orthogonal and are normed to 1, and hence, $\{\mathrm{e}^{+\mathrm{i}nx}\}_{n\in\mathbb{Z}}$ is an orthonormal system [1].

(ii) Since $f(x) \overset{[1]}{=} \frac{1}{\mathrm{i}2}\left(\mathrm{e}^{+\mathrm{i}2x} - \mathrm{e}^{-2x}\right)$ it lies in E, and f is its own Element of Best Approximation [1]. For g we proceed to compute the Fourier components, starting with $n = 0$:

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, x \overset{[1]}{=} \pi$$

For $n \neq 0$ we obtain

$$\begin{aligned}c_n \overset{[1]}{=} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, x\, \mathrm{e}^{+\mathrm{i}nx} &= \frac{1}{\mathrm{i}2\pi n} x\, \mathrm{e}^{+\mathrm{i}nx}\Big|_0^{2\pi} - \frac{1}{\mathrm{i}2\pi n}\int_0^{2\pi} \mathrm{d}x\, 1\cdot e^{inx} \\ &= \frac{1}{\mathrm{i}n} - \frac{1}{2\pi}\left[\frac{1}{(\mathrm{i}n)^2}\, \mathrm{e}^{+\mathrm{i}nx}\right]_0^{2\pi} \overset{[1]}{=} \frac{1}{\mathrm{i}n}\end{aligned}$$

Thus, the Element of Best Approximation of g in E is

$$g_E(x) = \pi + \sum_{n=1}^{4} \frac{1}{\mathrm{i}n}\left(\mathrm{e}^{+\mathrm{i}nx} - \mathrm{e}^{-\mathrm{i}nx}\right) \overset{[1]}{=} \pi + \sum_{n=1}^{4} \frac{2}{n}\sin nx.$$

(iii) The functions $\mathrm{e}^{+\mathrm{i}nx} \notin L^2(\mathbb{R})$ are not square integrable [1], and hence, the arguments cannot be repeated.

11. Multiplication operators (14 points)

Let $V \in L^\infty(\mathbb{R}^d)$ and define the multiplication operator

$$(T_V \psi)(x) := V(x)\,\psi(x)\,, \qquad\qquad \psi \in L^2(\mathbb{R}^d)\,.$$

(i) Show that $T_V : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is bounded.

(ii) Assume $V \in L^\infty(\mathbb{R}^d)$ is real-valued. Show that then $\langle \varphi, T_V \psi \rangle_{L^2(\mathbb{R}^d)} = \langle T_V \varphi, \psi \rangle_{L^2(\mathbb{R}^d)}$ holds for all $\varphi, \psi \in L^2(\mathbb{R}^d)$, i. e. T_V is selfadjoint.

(iii) Assume that V is bounded away from 0 and $+\infty$, i. e. that there exist $C > c > 0$ so that

$$0 < c \leq V(x) \leq C < +\infty$$

holds for all $x \in \mathbb{R}^d$. Find the inverse of T_V and show the inverse is bounded.

Solution:

(i) From the elementary estimate $\big|(T_V\psi)(x)\big| = \big|V(x)\,\psi(x)\big| \leq \|V\|_\infty \, \big|\psi(x)\big|$ [1], we deduce

$$\begin{aligned}
\big\|T_V\psi\big\| &\overset{[1]}{=} \left(\int_{\mathbb{R}^d} \mathrm{d}x \, \big|(T_V\psi)(x)\big|^2 \right)^{1/2} \\
&\overset{[1]}{\leq} \left(\int_{\mathbb{R}^d} \mathrm{d}x \, \|V\|_\infty^2 |\psi(x)|^2 \right)^{1/2} \\
&= \|V\|_\infty \left(\int_{\mathbb{R}^d} \mathrm{d}x \, |\psi(x)|^2 \right)^{1/2} \\
&\overset{[1]}{=} \|V\|_\infty \, \|\psi\|\,.
\end{aligned}$$

Hence, T_V is bounded [1].

(ii) The claim follows from $\overline{V} = V$ and direct computation: for any $\varphi, \psi \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
\langle \varphi, T_V\psi \rangle &\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{\varphi(x)} \, (T_V\psi)(x) \overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{\varphi(x)} \, V(x)\, \psi(x) \\
&= \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{V(x)\,\varphi(x)} \, \psi(x) = \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{(T_V\varphi)(x)} \, \psi(x) \\
&\overset{[1]}{=} \langle T_V \varphi, \psi \rangle\;.
\end{aligned}$$

(iii) Since V is bounded away from 0 and $+\infty$, so is V^{-1} [1],

$$0 < C^{-1} \leq V^{-1}(x) \leq c^{-1} < \infty\,.$$

Hence, also $T_{V^{-1}} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is a bounded multiplication operator by (i) [1]. Moreover, by direct computation, we verify that $T_{V^{-1}}$ is the inverse to T_V [1], e. g.

$$\begin{aligned}
\big(T_V \, T_{V^{-1}}\psi\big)(x) &\overset{[1]}{=} V(x)\,\big(T_{V^{-1}}\psi\big)(x) \\
&= V(x)\,V^{-1}(x)\,\psi(x) \overset{[1]}{=} \psi(x)\,,
\end{aligned}$$

and similarly $T_{V^{-1}}\,T_V = \mathrm{id}_{L^2(\mathbb{R}^d)}$ [1].