UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 3
(2013.09.26)

Stability of ODEs & Hamilton's Equations of Motion

Homework Problems

8. The Lipschitz property (7 points)

Let $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ be a function so that f' is bounded, i. e. there exists $L > 0$ such that

$$\sup_{x \in \mathbb{R}} \bigl| f'(x) \bigr| = L < \infty$$

holds for all $x \in \mathbb{R}$. Show that f is globally Lipschitz. (Hint: use the mean value theorem.)

Solution:

We need to show there exists $L > 0$ so that $\bigl| f(x) - f(y) \bigr| \leq L \, \bigl| x - y \bigr|$ holds for all $x, y \in \mathbb{R}$ [1]. For $x = y$, this statement is always satisfied [1]. So assume $x \neq y$.

We use the mean value theorem: for all $x, y \in \mathbb{R}$ with $x < y$ there exists a $c \in (x, y)$ so that

$$\frac{f(x) - f(y)}{x - y} = f'(c) \qquad [2]$$

holds. After taking the absolute value on both sides and using that f' is bounded, we obtain

$$\left| \frac{f(x) - f(y)}{x - y} \right| \overset{[1]}{=} \bigl| f'(c) \bigr| \overset{[1]}{\leq} \sup_{x \in \mathbb{R}} \bigl| f'(x) \bigr| = L.$$

Multiplying with $x - y$ on both sides yields the Lipschitz property,

$$\bigl| f(x) - f(y) \bigr| \leq L \, \bigl| x - y \bigr|. \qquad [1]$$

9. Hamiltonian equations of motion (23 points)

Let $H(q,p) = \frac{1}{2m}p^2 + V(q)$ be the energy function for a particle in one dimension subjected to the potential $V(q) = q^2 + \sin(\pi q)$, and consider Hamilton's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} +\partial_p H \\ -\partial_q H \end{pmatrix} =: X_H. \tag{1}$$

(i) Show that the Hamiltonian flow associated to (1) exists globally in time and for all initial conditions $(q_0, p_0) \in \mathbb{R}^2$.

(ii) Compute all fixed points of the Hamiltonian vector field X_H.

(iii) Investigate the stability of (1) around the fixed point as in Section 2.4, i. e. determine whether (1) is stable, Liapunov stable or unstable. Moreover, are the fixed points hyperbolic or elliptic?

(iv) Sketch the potential V and mark the fixed points as well as their stability properties.

Solution:

(i) We will show that

$$X_H(q,p) = \begin{pmatrix} +\partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} \frac{p}{m} \\ -2q - \pi\,\cos(\pi q) \end{pmatrix}$$

is globally Lipschitz and then invoke Corollary 2.2.8. With the exception of the term involving the cos, the terms are linear in either p or q, and thus with the help of

$$\begin{aligned} \lvert q - q' \rvert &\leq \left\lvert \begin{pmatrix} q \\ p \end{pmatrix} - \begin{pmatrix} q' \\ p' \end{pmatrix} \right\rvert && [1] \\ \lvert p - p' \rvert &\leq \left\lvert \begin{pmatrix} q \\ p \end{pmatrix} - \begin{pmatrix} q' \\ p' \end{pmatrix} \right\rvert \end{aligned}$$

we obtain global Lipschitz estimates for the first two of three terms,

$$\begin{aligned} \bigl\lvert X_H(q,p) - X_H(q',p') \bigr\rvert &\overset{[1]}{\leq} \left\lvert \begin{pmatrix} \frac{p}{m} \\ -2q - \pi\,\cos(\pi q) \end{pmatrix} - \begin{pmatrix} \frac{p'}{m} \\ -2q' - \pi\,\cos(\pi q') \end{pmatrix} \right\rvert \\ &\overset{[1]}{\leq} \left\lvert \tfrac{p}{m} - \tfrac{p'}{m} \right\rvert + \bigl\lvert -2q - (-2q') \bigr\rvert + \bigl\lvert \pi\,\cos(\pi q) - \pi\,\cos(\pi q') \bigr\rvert \\ &\leq \frac{1}{m} \left\lvert \begin{pmatrix} q \\ p \end{pmatrix} - \begin{pmatrix} q' \\ p' \end{pmatrix} \right\rvert + 2 \left\lvert \begin{pmatrix} q \\ p \end{pmatrix} - \begin{pmatrix} q' \\ p' \end{pmatrix} \right\rvert + \pi \bigl\lvert \cos(\pi q) - \pi\,\cos(\pi q') \bigr\rvert \\ &\overset{[1]}{=} \bigl(\tfrac{1}{m} + 2\bigr) \left\lvert \begin{pmatrix} q \\ p \end{pmatrix} - \begin{pmatrix} q' \\ p' \end{pmatrix} \right\rvert + \pi \bigl\lvert \cos(\pi q) - \pi\,\cos(\pi q') \bigr\rvert . \end{aligned}$$

To obtain a Lipschitz estimate for the last term, we use problem 8 [1]: since the derivative of $\pi\,\cos(\pi q)$ is bounded, it is Lipschitz, and the smallest Lipschitz constant is

$$\sup_{q \in \mathbb{R}} \lvert \pi\,\partial_q \cos(\pi q) \rvert = \sup_{q \in \mathbb{R}} \bigl\lvert \pi^2\,\sin(\pi q) \bigr\rvert = \pi^2 .$$

Hence, the vector field is globally Lipschitz,

$$\bigl\lvert X_H(q,p) - X_H(q',p') \bigr\rvert \overset{[1]}{\leq} \bigl(\tfrac{1}{m} + 2 + \pi^2\bigr) \left\lvert \begin{pmatrix} q \\ p \end{pmatrix} - \begin{pmatrix} q' \\ p' \end{pmatrix} \right\rvert \qquad \forall (q,p), (q',p') \in \mathbb{R}^2,$$

and thus by Corollary 2.2.8, the Hamiltonian flow Φ exists globally in time [1].

(ii) Fixed points (q_0, p_0) are stationary points of the vector field,

$$X_H(q_0, p_0) = \begin{pmatrix} \frac{p_0}{m} \\ -V'(q_0) \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies $p_0 = 0$ and $V'(q_0) = 0$ [1]. V has three local extrema, $q_{0\,1} \approx -0.42$, $q_{0\,2} \approx 0.63$ and $q_{0\,3} \approx 1.22$ [3].

(iii) The differential of the Hamiltonian vector field is

$$DX_H(q,p) = \begin{pmatrix} 0 & 1/m \\ -V''(q) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ -2 + \pi^2 \sin(\pi q) & 0 \end{pmatrix} \qquad [1].$$

This matrix has the eigenvalues

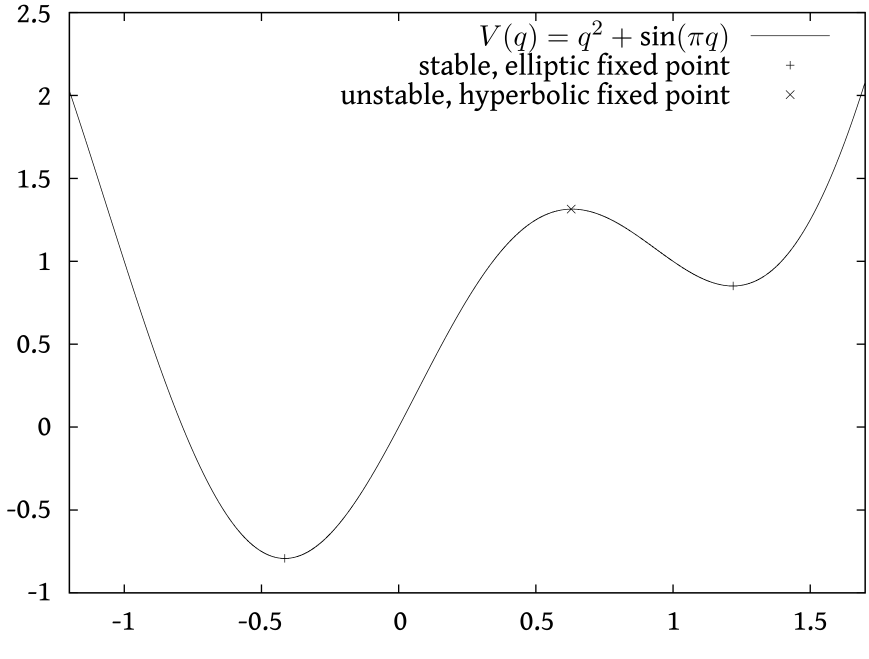
$$\lambda_\pm(q) = \pm\sqrt{\frac{\pi^2 \sin(\pi q) - 2}{m}}. \qquad [1]$$

$q_{0\,1} \approx -0.42$: The term under the square root is negative and thus the $\lambda_\pm \in \mathrm{i}\mathbb{R}$ are purely imaginary. Thus, the vector field is marginally stable [1], elliptic [1] and not hyperbolic.

$q_{0\,2} \approx 0.63$: The term under the square root is positive and thus the $\lambda_\pm \in \mathbb{R}$ are purely real with $\lambda_+ > 0 > \lambda_-$. Thus, the vector field is unstable [1], hyperbolic [1] and not elliptic.

$q_{0\,3} \approx 1.22$: The term under the square root is negative and thus the $\lambda_\pm \in \mathrm{i}\mathbb{R}$ are purely imaginary. Thus, the vector field is marginally stable [1], elliptic [1] and not hyperbolic.

(iv) The stable, elliptic fixed points correspond to local minima while the local maximum is unstable and hyperbolic. [3]



10. A two-dimensional classical particle in a magnetic field (17 points)

Assume the magnetic field $b \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ is smooth and bounded, and define the associated magnetic field matrix

$$B(q) = \begin{pmatrix} 0 & -b(q) \\ +b(q) & 0 \end{pmatrix}$$

Moreover, let $H(q,p) = \frac{1}{2}p^2$ be the energy function for a particle with mass 1 and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & +\mathrm{id}_{\mathbb{R}^2} \\ -\mathrm{id}_{\mathbb{R}^2} & B \end{pmatrix} \begin{pmatrix} \nabla_q H \\ \nabla_p H \end{pmatrix} =: X_H \tag{2}$$

its equations of motion.

(i) Find the fixed points of the Hamiltonian vector field X_H and investigate the stability properties of (2) at those fixed points.

(ii) Now assume b is constant. Solve the equations of motion explicitly for the initial conditions (q_0, p_0). (You may make use of all your previous homework problems.)

Solution:

(i) The fixed points of the Hamiltonian vector field

$$\begin{aligned} X_H(q,p) &= \begin{pmatrix} 0 & +\mathrm{id}_{\mathbb{R}^2} \\ -\mathrm{id}_{\mathbb{R}^2} & B(q) \end{pmatrix} \begin{pmatrix} \nabla_q H(q,p) \\ \nabla_p H(q,p) \end{pmatrix} \\ &= \begin{pmatrix} 0 & +\mathrm{id}_{\mathbb{R}^2} \\ -\mathrm{id}_{\mathbb{R}^2} & B(q) \end{pmatrix} \begin{pmatrix} 0 \\ p \end{pmatrix} \\ &\overset{[2]}{=} \begin{pmatrix} p \\ B(q)p \end{pmatrix} \end{aligned}$$

are $(q_0, 0)$ [1].

We then compute the differential

$$DX_H(q,p) \overset{[1]}{=} \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^2} \\ B'(q)p & B(q) \end{pmatrix}.$$

and set $p = 0$,

$$DX_H(q_0, 0) \overset{[1]}{=} \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^2} \\ 0 & B(q_0) \end{pmatrix}.$$

Since this is a block matrix, the eigenvalues can be computed easily using

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \, \det D,$$

because then

$$\begin{aligned} \det\big(\lambda \, \mathrm{id}_{\mathbb{C}^4} - DX_H(q_0, 0)\big) &\overset{[1]}{=} \det \begin{pmatrix} \lambda \, \mathrm{id}_{\mathbb{C}^2} & -\mathrm{id}_{\mathbb{C}^2} \\ 0 & \lambda \, \mathrm{id}_{\mathbb{C}^2} - B(q_0) \end{pmatrix} \\ &= \det\big(\lambda \, \mathrm{id}_{\mathbb{C}^2}\big) \, \det\big(\lambda \, \mathrm{id}_{\mathbb{C}^2} - B(q_0)\big) \\ &= \lambda^2 \det \begin{pmatrix} \lambda & +b(q_0) \\ -b(q_0) & \lambda \end{pmatrix} \\ &\overset{[1]}{=} \lambda^2 \left(\lambda^2 + b(q_0)^2\right). \end{aligned}$$

The eigenvalues of this matrix are thus $\lambda_{1,2} = 0$ [1] and $\lambda_{3,4} = \pm \mathrm{i} \sqrt{|b(q_0)|}$ [1]. This means *all* fixed points are *marginally stable* [1] and *elliptic* [1].

(ii) Now the magnetic field b is constant. Given that $\dot{q}(t) = p(t)$, we can obtain $q(t)$ by integrating $p(t)$,

$$q(t) \overset{[1]}{=} q_0 + \int_0^t \mathrm{d}s \, p(s).$$

So let us solve the equation for $p(t)$ which just a linear ODE:

$$\dot{p}(t) \overset{[1]}{=} B\,p = \begin{pmatrix} 0 & -b \\ +b & 0 \end{pmatrix} p$$

We can solve this by using problem 3 from sheet 1, because B is just the lower 2×2 block of the matrix H of problem 3. Thus the matrix exponential is also just the lower 2×2 matrix block, i. e.

$$\begin{aligned} p(t) &\overset{[1]}{=} \mathrm{e}^{tB} p_0 = \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix} \begin{pmatrix} p_{0\,1} \\ p_{0\,2} \end{pmatrix} \\ &\overset{[1]}{=} \begin{pmatrix} p_{0\,1} \cos(bt) - p_{0\,2} \sin(bt) \\ p_{0\,1} \sin(bt) + p_{0\,2} \cos(bt) \end{pmatrix}. \end{aligned}$$

So we can integrate $p(t)$ explicitly, and obtain

$$\begin{aligned} q(t) &\overset{[1]}{=} \begin{pmatrix} q_{0\,1} \\ q_{0\,2} \end{pmatrix} + \int_0^t \mathrm{d}s \begin{pmatrix} p_{0\,1} \cos(bs) - p_{0\,2} \sin(bs) \\ p_{0\,1} \sin(bs) + p_{0\,2} \cos(bs) \end{pmatrix} \\ &= \begin{pmatrix} q_{0\,1} \\ q_{0\,2} \end{pmatrix} + \left[\frac{1}{b} \begin{pmatrix} p_{0\,1} \sin(bs) + p_{0\,2} \cos(bs) \\ -p_{0\,1} \cos(bs) + p_{0\,2} \sin(bs) \end{pmatrix} \right]_0^t \\ &\overset{[1]}{=} \begin{pmatrix} q_{0\,1} \\ q_{0\,2} \end{pmatrix} + \frac{1}{b} \begin{pmatrix} p_{0\,1} \sin(bt) + p_{0\,2} \big(\cos(bt) - 1\big) \\ -p_{0\,1} \big(\cos(bt) - 1\big) + p_{0\,2} \sin(bt) \end{pmatrix} \end{aligned}$$