UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics (APM 421 H)

Winter 2014
Problem Sheet 3
(2014.09.26)

Hilbert Spaces & Operators

Homework Problems

8. Orthogonal subspaces and projections onto subspaces

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis (ONB) of a Hilbert space $\mathcal{H}$ and $N \in \mathbb{N}$.

(i) Prove that $E := \big\{\varphi_1, \ldots, \varphi_N\big\}^{\perp}$ is a sub *vector* space.

(ii) Give an ONB for the subspace $E = \big\{\varphi_1, \ldots, \varphi_N\big\}^{\perp}$.

(iii) Show that $\big(\{\varphi_1, \ldots, \varphi_N\}^{\perp}\big)^{\perp} = E^{\perp} = \mathrm{span}\big\{\varphi_1, \ldots, \varphi_N\big\}$.

Moreover, define the map

$$P : \mathcal{H} \longrightarrow \mathcal{H}\,,\ P\psi := \sum_{n=1}^{N} \langle\varphi_n, \psi\rangle\ \varphi_n\,.$$

(iv) Show that P is linear, i. e. for any $\varphi, \psi \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, we have $P\big(\alpha\varphi + \psi\big) = \alpha\, P\varphi + P\psi$.

(v) Show that P is a projection, i. e. $P^2 = P$.

(vi) Show that P is bounded, i. e. $\|P\varphi\| \leq \|\varphi\|$ holds for any $\varphi \in \mathcal{H}$.

9. The Fock space

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ the n-fold tensor product. By definition we set $\mathcal{H}^{\otimes 0} := \mathbb{C}$. Define the *Fock space* over $\mathcal{H}$ as $\mathfrak{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$ with scalar product

$$\big\langle\varphi, \psi\big\rangle_{\mathfrak{F}} := \sum_{n=0}^{\infty} \big\langle\varphi_n, \psi_n\big\rangle_{\mathcal{H}^{\otimes n}}, \qquad \varphi = \big(\varphi_0, \varphi_1, \ldots\big), \psi = \big(\psi_0, \psi_1, \ldots\big) \in \mathfrak{F}(\mathcal{H}).$$

In the physics literature, $\mathcal{H}^{\otimes n}$ is called the *n-particle Fock sector*.

(i) Give a countable orthonormal basis of $\mathfrak{F}(\mathcal{H})$. (No proof is necessary.)

(ii) Show that for any element $\psi = \big(\psi_0, \psi_1, \psi_2, \ldots\big)$ of $\mathfrak{F}(\mathcal{H})$

$$\lim_{n\to\infty} \big\|\psi_n\big\|_{\mathcal{H}^{\otimes n}} = 0$$

holds where ψ_n is the corresponding element of the n-particle Fock sector.

Let A be a linear, bounded operator on $\mathcal{H}$ with domain $\mathcal{D}(A) = \mathcal{H}$. Then we define the *second quantization of A* as the operator

$$\mathrm{d}\Gamma(A) := \bigoplus_{n=0}^{\infty} \big(A \otimes \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} + \ldots + \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} \otimes A\big)$$

acting on $\mathfrak{F}(\mathcal{H})$.

(iii) Give the action of $\mathrm{d}\Gamma(A)$ on the nth Fock sector $\mathcal{H}^{\otimes n}$, i. e. find $\big(\mathrm{d}\Gamma(A)\psi\big)_n$.

(iv) Give the domain of $\mathrm{d}\Gamma(\mathrm{id}_{\mathcal{H}})$ and discuss the physical meaning of $\mathrm{d}\Gamma(\mathrm{id}_{\mathcal{H}})$.

10. Best approximation: Fourier series

Let $L^2([0, 2\pi])$ be the Hilbert space of square-integrable functions with scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}x \, \overline{f(x)} \, g(x).$$

(i) Show that $\{\mathrm{e}^{+inx}\}_{n \in \mathbb{Z}}$ is an orthonormal system of vectors in $L^2([0, 2\pi])$.

(ii) Let $E := \{\mathrm{e}^{+inx}\}_{\substack{n \in \mathbb{Z} \\ |n| \leq 4}}$, and consider the functions $f(x) = \sin(2x)$ and $g(x) = x$. Give the element of best approximation of f and g in span E.

(iii) Why don't these arguments work for $L^2(\mathbb{R})$?

11. Multiplication operators

Let $V \in L^\infty(\mathbb{R}^d)$ and define the multiplication operator

$$(T_V \psi)(x) := V(x) \, \psi(x) \, , \qquad \psi \in L^2(\mathbb{R}^d) \, .$$

(i) Show that $T_V : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is bounded.

(ii) Assume $V \in L^\infty(\mathbb{R}^d)$ is real-valued. Show that then $\langle \varphi, T_V \psi \rangle_{L^2(\mathbb{R}^d)} = \langle T_V \varphi, \psi \rangle_{L^2(\mathbb{R}^d)}$ holds for all $\varphi, \psi \in L^2(\mathbb{R}^d)$, i. e. T_V is selfadjoint.

(iii) Assume that V is bounded away from 0 and $+\infty$, i. e. that there exist $C > c > 0$ so that

$$0 < c \leq V(x) \leq C < +\infty$$

holds for all $x \in \mathbb{R}^d$. Find the inverse of T_V and show the inverse is bounded.

Hand in home work on: Friday, 3 October 2014, before class