UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 3
(2013.09.26)

Stability of ODEs & Hamilton's Equations of Motion

Homework Problems

8. The Lipschitz property (7 points)

Let $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ be a function so that f' is bounded, i. e. there exists $L > 0$ such that

$$\sup_{x \in \mathbb{R}} \bigl| f'(x) \bigr| = L < \infty$$

holds for all $x \in \mathbb{R}$. Show that f is globally Lipschitz. (Hint: use the mean value theorem.)

9. Hamiltonian equations of motion (23 points)

Let $H(q,p) = \frac{1}{2m} p^2 + V(q)$ be the energy function for a particle in one dimension subjected to the potential $V(q) = q^2 + \sin(\pi q)$, and consider Hamilton's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} +\partial_p H \\ -\partial_q H \end{pmatrix} =: X_H . \tag{1}$$

(i) Show that the Hamiltonian flow associated to (1) exists globally in time and for all initial conditions $(q_0, p_0) \in \mathbb{R}^2$.

(ii) Compute all fixed points of the Hamiltonian vector field X_H.

(iii) Investigate the stability of (1) around the fixed point as in Section 2.4, i. e. determine whether (1) is stable, Liapunov stable or unstable. Moreover, are the fixed points hyperbolic or elliptic?

(iv) Sketch the potential V and mark the fixed points as well as their stability properties.

10. A two-dimensional classical particle in a magnetic field (17 points)

Assume the magnetic field $b \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ is smooth and bounded, and define the associated magnetic field matrix

$$B(q) = \begin{pmatrix} 0 & -b(q) \\ +b(q) & 0 \end{pmatrix}$$

Moreover, let $H(q,p) = \frac{1}{2} p^2$ be the energy function for a particle with mass 1 and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & +\mathrm{id}_{\mathbb{R}^2} \\ -\mathrm{id}_{\mathbb{R}^2} & B \end{pmatrix} \begin{pmatrix} \nabla_q H \\ \nabla_p H \end{pmatrix} =: X_H \tag{2}$$

its equations of motion.

(i) Find the fixed points of the Hamiltonian vector field X_H and investigate the stability properties of (2) at those fixed points.

(ii) Now assume b is constant. Solve the equations of motion explicitly for the initial conditions (q_0, p_0). (You may make use of all your previous homework problems.)

Hand in home work on: Thursday, 3 October 2013, before class