UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 4
(2013.10.03)

Classical Mechanics

Homework Problems

11. The Gaußian integral (4 points)

Show that

$$\int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

Solution:

We compute the square and use polar coordinates:

$$\begin{aligned}\left(\int_{\mathbb{R}} \mathrm{d}x \, e^{-ax^2}\right)^2 &\overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y \, \mathrm{e}^{-a(x^2+y^2)} \\ &\overset{[1]}{=} \int_0^{\infty} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi \, r \, e^{-ar^2} \\ &= 2\pi \left[-\frac{1}{2a}\mathrm{e}^{-ar^2}\right]_0^{\infty} \overset{[1]}{=} \frac{\pi}{a}\end{aligned}$$

Taking the square root yields the required equation,

$$\int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{-ax^2} \overset{[1]}{=} \sqrt{\frac{\pi}{a}}.$$

12. Angular momentum as generator of rotations (11 points)

Consider the angular momentum observable $L(q,p) = \big(L_1(q,p), L_2(q,p), L_3(q,p)\big) := q \times p$. Show that L generates rotations:

(i) Solve

$$\frac{\mathrm{d}}{\mathrm{d}\omega} q(\omega) = \big\{L_1, q(\omega)\big\}, \qquad q(0) = q_0 \in \mathbb{R}^3,$$

explicitly.

(ii) Give the solution to

$$\frac{\mathrm{d}}{\mathrm{d}\omega} p(\omega) = \big\{L_1, p(\omega)\big\}, \qquad p(0) = p_0 \in \mathbb{R}^3,$$

explicitly. (You need not calculate the same thing twice.)

(iii) Give the flow Ψ to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \{L_1, q\} \\ \{L_1, p\} \end{pmatrix}.$$

Does Ψ exist for all $\omega \in \mathbb{R}$?

Solution:

(i) Corollary 3.3.5 tells us we can compute $\{L_1, q\}$ instead of $\big\{L_1, q(\omega)\big\}$ [1], because

$$\frac{\mathrm{d}}{\mathrm{d}\omega} q(\omega) = \big\{L_1, q(\omega)\big\} \overset{[1]}{=} \{L_1, q\} \circ \Psi_\omega.$$

The Poisson bracket can be computed as

$$\frac{\mathrm{d}}{\mathrm{d}\omega} q_j = \{L_1, q_j\} \overset{[1]}{=} \partial_{p_j} \big(q_2\, p_3 - q_3\, p_2\big).$$

Hence, combining the three components yields the equation of motion of a classical spin (sheet 1, problem 3),

$$\frac{\mathrm{d}}{\mathrm{d}\omega} q = \begin{pmatrix} 0 \\ -q_3 \\ +q_2 \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} =: \mathcal{L}\, q,$$

and thus the solution to $q(0) = q_0$ is given in terms of the matrix exponential $\mathrm{e}^{\omega \mathcal{L}}$ and the initial condition q_0,

$$q(t) \overset{[1]}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix} q_0 =: R(\omega)\, q_0.$$

(ii) Similarly, we obtain the equations of motion for the momenta,

$$\frac{\mathrm{d}}{\mathrm{d}\omega} p_j = \{L_1, p_j\} \overset{[1]}{=} -\partial_{q_j} \big(q_2\, p_3 - q_3\, p_2\big),$$

i. e. collecting all three components, we once again recover the same equations as in (i):

$$\frac{\mathrm{d}}{\mathrm{d}\omega}p = \begin{pmatrix} 0 \\ -p_3 \\ +p_2 \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \mathcal{L}\, p.$$

The solution is again

$$p(\omega) \overset{[1]}{=} R(\omega)\, p_0$$

where $R(\omega)$ is the same rotation matrix as in (i).

(iii) Now the flow can be easily expressed in terms of $R(\omega)$,

$$\Psi_\omega(q, p) \overset{[2]}{=} \big(R(\omega)\, q, R(\omega)\, p\big).$$

Ψ exists for all $\omega \in \mathbb{R}$ [1].

13. Averages with respect to states & the spectrum of observables (20 points)

(i) Show that the Gaußian measure

$$\mu_{a,b}(A) := \frac{1}{\pi ab} \int_A \mathrm{d}q \, \mathrm{d}p \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}} , \qquad A \subset \mathbb{R}^2 \text{ Borel set,}$$

localized around a point in phase space $(q_0, p_0) \in \mathbb{R}^2$ of widths $a, b > 0$ is a classical state in the sense of Definition 3.1.1. (You need not prove that $\mu_{a,b}$ is a Borel measure.)

(ii) Compute the energy average

$$\mathbb{E}_{\mu_{a,b}}(H) = \frac{1}{\pi ab} \int_{\mathbb{R}^2} \mathrm{d}q \, \mathrm{d}p \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}} \, H(q,p)$$

for the *one-dimensional* harmonic oscillator Hamiltonian $H(q,p) = \frac{1}{2}(p^2 + q^2)$ with respect to the Gaußian state $\mu_{a,b}$.

(iii) Show that $\lim_{a,b \to 0} \mathbb{E}_{\mu_{a,b}}(H) = H(q_0, p_0)$.

(iv) Now consider the case where phase space is $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. Show that the each of the three components of angular momentum $L(q,p) = q \times p$ are constants of motion for the *three-dimensional* harmonic oscillator dynamics generated by $H_{\mathbb{R}^3}(q,p) := \sum_{j=1}^3 H(q_j, p_j)$.

(v) Give the spectrum for the observables q_1, p_1, L_1 and H.

Solution:

(i) *Positivity:* This follows directly from the positivity of the Gaußian,

$$\frac{1}{\pi ab} \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}} > 0, \qquad [1]$$

so that also

$$\mu_{a,b}(A) = \frac{1}{\pi ab} \int_A \mathrm{d}q \, \mathrm{d}p \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}} > 0 \qquad [1]$$

holds true for any Borel set $A \subseteq \mathbb{R}^2$.

Normalization: We now use the fact that the integral factors as well as problem 11:

$$\begin{aligned}
\mu_{a,b}(\mathbb{R}^2) &\overset{[1]}{=} \frac{1}{\pi ab} \int_{\mathbb{R}^2} \mathrm{d}q \, \mathrm{d}p \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}} \\
&= \frac{1}{\pi ab} \left(\int_{\mathbb{R}} \mathrm{d}q \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \right) \left(\int_{\mathbb{R}} \mathrm{d}p \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}} \right) \\
&\overset{[1]}{=} \frac{1}{\pi ab} \left(\sqrt{\pi} \, a \right) \left(\sqrt{\pi} \, b \right) \overset{[1]}{=} 1
\end{aligned}$$

Hence, the measure is also normalized.

(ii) Since the expectation value is linear, we can rewrite the expectation value as the sum of two similar terms:

$$\mathbb{E}_{\mu_{a,b}}(H) \overset{[1]}{=} \tfrac{1}{2} \big(\mathbb{E}_{\mu_{a,b}}(p^2) + \mathbb{E}_{\mu_{a,b}}(q^2) \big)$$

Given the symmetry, we will only calculate $\mathbb{E}_{\mu_{a,b}}(p^2)$ in detail:

$$\begin{aligned}
\mathbb{E}_{\mu_{a,b}}(p^2) &\overset{[1]}{=} \frac{1}{\pi ab} \int_{\mathbb{R}^2} \mathrm{d}q\,\mathrm{d}p\; \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}}\; \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}}\; p^2 \\
&= \frac{1}{\pi ab} \left(\int_{\mathbb{R}} \mathrm{d}q\; \mathrm{e}^{-\frac{q^2}{a^2}} \right) \left(\int_{\mathbb{R}} \mathrm{d}p\; (p+p_0)^2\, \mathrm{e}^{-\frac{p^2}{b^2}} \right) \\
&\overset{[1]}{=} \frac{1}{\pi ab} \left(\sqrt{\pi}\, a\right) \int_{\mathbb{R}} \mathrm{d}p\; \left(p^2 + 2p_0\, p + p_0^2\right) \mathrm{e}^{-\frac{p^2}{b^2}} \\
&= \frac{1}{\sqrt{\pi}\, b} \int_{\mathbb{R}} \mathrm{d}p\; \left(p^2 + 2p_0\, p + p_0^2\right) \mathrm{e}^{-\frac{p^2}{b^2}}
\end{aligned}$$

The second and third term of the integral can be computed directly using the symmetry $p \mapsto -p$ of the integrand:

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}p\; 2p_0\, p\, \mathrm{e}^{-\frac{p^2}{b^2}} &\overset{[1]}{=} 2 \int_0^\infty \mathrm{d}p\; 2p_0\, p\, \mathrm{e}^{-\frac{p^2}{b^2}} \\
&= 4p_0 \left[-\frac{b^2}{2}\, \mathrm{e}^{-\frac{p^2}{b^2}} \right]_0^\infty \overset{[1]}{=} 2p_0\, b^2 \\
\int_{\mathbb{R}} \mathrm{d}p\; p_0^2\, \mathrm{e}^{-\frac{p^2}{b^2}} &= p_0^2 \sqrt{\pi}\, b
\end{aligned}$$

We use the symmetry $p \mapsto -p$ of the integrand again and apply partial integration to compute the first term:

$$\begin{aligned}
\frac{1}{\sqrt{\pi}\, b} \int_{\mathbb{R}} \mathrm{d}p\; p^2\, \mathrm{e}^{-\frac{p^2}{b^2}} &\overset{[1]}{=} \frac{2}{\sqrt{\pi}\, b} \int_0^\infty \mathrm{d}p\; \underbrace{p}_{=u}\; \underbrace{p\, \mathrm{e}^{-\frac{p^2}{b^2}}}_{=v'} \\
&= \frac{2}{\sqrt{\pi}\, b} \left[-\frac{b^2}{2}\, p\, \mathrm{e}^{-\frac{p^2}{b^2}} \right]_0^\infty + \frac{2}{\sqrt{\pi}\, b} \frac{b^2}{2} \int_0^\infty \mathrm{d}p\, \mathrm{e}^{-\frac{p^2}{b^2}} \\
&\overset{[1]}{=} \frac{b^2}{2}
\end{aligned}$$

Overall, we obtain

$$\begin{aligned}
\mathbb{E}_{\mu_{a,b}}(p^2) &= \frac{1}{\sqrt{\pi}\, b} \left(\frac{b^2}{2} + 2p_0\, b^2 + p_0^2 \sqrt{\pi}\, b \right) \\
&\overset{[1]}{=} \frac{b}{2\sqrt{\pi}} + \frac{2p_0\, b}{\sqrt{\pi}} + p_0^2.
\end{aligned}$$

Exchanging the roles of q and p as well as of a and b yields

$$\mathbb{E}_{\mu_{a,b}}(q^2) = \frac{a}{2\sqrt{\pi}} + \frac{2q_0\, a}{\sqrt{\pi}} + q_0^2,$$

and thus

$$\mathbb{E}_{\mu_{a,b}}(H) \overset{[1]}{=} \frac{a}{4\sqrt{\pi}} + \frac{q_0\, a}{\sqrt{\pi}} + \frac{q_0^2}{2} + \frac{b}{4\sqrt{\pi}} + \frac{p_0\, b}{\sqrt{\pi}} + \frac{p_0^2}{2}.$$

(iii) Since $\mathbb{E}_{\mu_{a,b}}(H)$ is a quadratic polynomial in a and b, taking the limit is trivial, and we obtain

$$\begin{aligned}
\lim_{a,b\to 0} \mathbb{E}_{\mu_{a,b}}(H) &= \lim_{a,b\to 0} \left(\frac{a}{4\sqrt{\pi}} + \frac{q_0\, a}{\sqrt{\pi}} + \frac{q_0^2}{2} + \frac{b}{4\sqrt{\pi}} + \frac{p_0\, b}{\sqrt{\pi}} + \frac{p_0^2}{2} \right) \\
&\overset{[1]}{=} \tfrac{1}{2}\left(p_0^2 + q_0^2\right).
\end{aligned}$$

(iv) According to Corollary 3.3.5, an observable f is a constant of motion iff $\{H_{\mathbb{R}^3}, f\} = 0$.

The computations are straight-forward. For the first component we give the computation in full detail:

$$\begin{aligned}
\{H_{\mathbb{R}^3}, L_1\} &= \nabla_p H_{\mathbb{R}^3} \cdot \nabla_q L_1 - \nabla_q H_{\mathbb{R}^3} \cdot \nabla_p L_1 \\
&= p \cdot \nabla_q \big(q_2\, p_3 - q_3\, p_2\big) - q \cdot \nabla_p \big(q_2\, p_3 - q_3\, p_2\big) \\
&= p \cdot \begin{pmatrix} 0 \\ +p_3 \\ -p_2 \end{pmatrix} - q \cdot \begin{pmatrix} 0 \\ -q_3 \\ +q_2 \end{pmatrix} = \big(p_2\, p_3 - p_3\, p_2\big) - \big(-q_2\, q_3 + q_3\, q_2\big) \\
&\overset{[1]}{=} 0
\end{aligned}$$

The other components are computed similarly.

$$\begin{aligned}
\{H_{\mathbb{R}^3}, L_2\} &= p \cdot \begin{pmatrix} -p_3 \\ 0 \\ +p_1 \end{pmatrix} - q \cdot \begin{pmatrix} +q_3 \\ 0 \\ -q_1 \end{pmatrix} \overset{[1]}{=} 0 \\
\{H_{\mathbb{R}^3}, L_3\} &= p \cdot \begin{pmatrix} +p_2 \\ -p_1 \\ 0 \end{pmatrix} - q \cdot \begin{pmatrix} -q_2 \\ +q_1 \\ 0 \end{pmatrix} \overset{[1]}{=} 0
\end{aligned}$$

(v) The spectrum of the observable is the image of the function (cf. Definition 3.1.3):

$$\begin{aligned}
\text{spec}\, q_1 &= \text{im}\, q_1 \overset{[1/2]}{=} \mathbb{R} \\
\text{spec}\, p_1 &= \text{im}\, p_1 \overset{[1/2]}{=} \mathbb{R} \\
\text{spec}\, L_1 &= \text{im}\, L_1 \overset{[1/2]}{=} \mathbb{R} \\
\text{spec}\, H &= \text{im}\, H \overset{[1/2]}{=} [0, +\infty)
\end{aligned}$$

14. Magnetic classical systems (17 points)

Consider the equations of motion of a non-relativistic particle in three dimensions which is subjected to an electromagnetic field where $\mathbf{E} = -\nabla_q V$ is the electric field and $\mathbf{B} = \bigl(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\bigr)$. In other words, we are considering the Hamilton function H and the magnetic version of Hamilton's equations of motion

$$\begin{pmatrix} B & -\mathrm{id}_{\mathbb{R}^3} \\ +\mathrm{id}_{\mathbb{R}^3} & 0 \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \nabla_q H \\ \nabla_p H \end{pmatrix} \tag{1}$$

where the magnetic field matrix

$$B(q) = \begin{pmatrix} 0 & +\mathbf{B}_3 & -\mathbf{B}_2 \\ -\mathbf{B}_3 & 0 & +\mathbf{B}_1 \\ +\mathbf{B}_2 & -\mathbf{B}_1 & 0 \end{pmatrix}$$

is defined in terms of the components of $\mathbf{B}$. We denote the corresponding Hamiltonian flow with Φ.

Moreover, we define the *magnetic* Poisson bracket

$$\{f, g\}_B := \sum_{j=1}^{3} \bigl(\partial_{p_j} f \; \partial_{q_j} g - \partial_{q_j} f \; \partial_{p_j} g \bigr) - \sum_{j,k=1}^{3} B_{jk} \, \partial_{p_j} f \; \partial_{p_k} g .$$

(i) Show that $\{\cdot\,,\cdot\}_B$ generates equations (1).
(Hint: Consider the equations of motion generated by H for q and p in the Heisenberg picture.)

(ii) Show that $\mathbf{B}$ is source-free, i. e. $\nabla_q \cdot \mathbf{B} = 0$.
(Hint: Rewrite the magnetic field $\mathbf{B} = \nabla_q \times \mathbf{A}$ as the curl of a vector potential $\mathbf{A}$.)

(iii) Show that $\{f, g\}_B$ is antisymmetric and has the derivation property (cf. Proposition 3.3.4).

(iv) Show that energy is a constant of motion by computing the time-derivative of $H(t) := H \circ \Phi_t$.

Solution:

(i) First, let us start with the equations of motion for position:

$$\begin{aligned} \dot{q}_j &= \{H, q_j\}_B \overset{[1]}{=} \{H, q_j\} - \sum_{l,k=1}^{3} B_{lk} \, \partial_{p_l} H \, \partial_{p_k} q_j \\ &\overset{[1]}{=} \partial_{p_j} H \end{aligned}$$

The equations of motion for p contain a magnetic contribution:

$$\begin{aligned} \dot{p}_j &= \{H, q_j\}_B \overset{[1]}{=} \{H, p_j\} - \sum_{l,k=1}^{3} B_{lk} \, \partial_{p_l} H \, \partial_{p_k} p_j \\ &\overset{[1]}{=} -\partial_{q_j} H - \sum_{l=1}^{3} B_{lj} \, \partial_{p_l} H \overset{[1]}{=} -\partial_{q_j} H + \sum_{l=1}^{3} B_{jl} \, \dot{q}_j \end{aligned}$$

If we collect these equations for the components of $\dot{q}$ and $\dot{p}$, then we recover (1):

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} +\nabla_p H \\ -\nabla_q H + B \, \dot{q} \end{pmatrix}$$

$$\Leftrightarrow$$

$$\begin{pmatrix} B & -\mathrm{id}_{\mathbb{R}^3} \\ +\mathrm{id}_{\mathbb{R}^3} & 0 \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} \nabla_q H \\ \nabla_p H \end{pmatrix}$$

(ii) Let us denote the trajectory starting at (q_0, p_0) with $\big(q(t), p(t)\big) = \Phi_t(q_0, p_0)$. Then we compute:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(H(t)\big)(q_0,p_0) &= \frac{\mathrm{d}}{\mathrm{d}t} H\big(q(t),p(t)\big) \\
&\overset{[1]}{=} \sum_{j=1}^{3} \Big(\partial_{q_j} H\big(q(t),p(t)\big)\, \dot{q}_j(t) + \partial_{p_j} H\big(q(t),p(t)\big)\, \dot{p}_j(t) \Big) \\
&\overset{[1]}{=} \sum_{j=1}^{3} \Bigg(\partial_{q_j} H\big(q(t),p(t)\big)\, \partial_{p_j} H\big(q(t),p(t)\big) + \partial_{p_j} H\big(q(t),p(t)\big) \cdot \\
&\qquad \cdot \Bigg(-\partial_{q_j} H\big(q(t),p(t)\big) + \sum_{k=1}^{3} B_{jk}\big(q(t)\big)\, \partial_{p_k} H\big(q(t),p(t)\big) \Bigg) \Bigg) \\
&= \nabla_p H\big(q(t),p(t)\big) \cdot B\big(q(t)\big)\, \nabla_p H\big(q(t),p(t)\big) \\
&\overset{[1]}{=} \nabla_p H\big(q(t),p(t)\big) \cdot \Big(\nabla_p H\big(q(t),p(t)\big) \times \mathbf{B}\big(q(t)\big) \Big) \overset{[1]}{=} 0
\end{aligned}$$

Since the above holds for any choice of (q_0, p_0), we deduce that energy is conserved [1].

(iii) Any magnetic field $\mathbf{B} = \nabla_q \times \mathbf{A}$ can be written as the curl of a vector potential $\mathbf{A}$, and thus, using the standard equality $\nabla_q \cdot \nabla_q \times \mathbf{A} = 0$ [2], we deduce $\mathbf{B}$ is source-free:

$$\begin{aligned}
\nabla_q \cdot \mathbf{B} &= \nabla_q \cdot \big(\nabla_q \times \mathbf{A}\big) \\
&= \partial_{q_1}\big(\partial_{q_2}\mathbf{A}_3 - \partial_{q_3}\mathbf{A}_2\big) + \partial_{q_2}\big(\partial_{q_3}\mathbf{A}_1 - \partial_{q_1}\mathbf{A}_3\big) + \partial_{q_3}\big(\partial_{q_1}\mathbf{A}_2 - \partial_{q_2}\mathbf{A}_1\big) \\
&= \partial_{q_1}\partial_{q_2}\mathbf{A}_3 - \partial_{q_2}\partial_{q_1}\mathbf{A}_3 - \partial_{q_1}\partial_{q_3}\mathbf{A}_2 + \partial_{q_3}\partial_{q_1}\mathbf{A}_2 + \partial_{q_2}\partial_{q_3}\mathbf{A}_1 - \partial_{q_3}\partial_{q_2}\mathbf{A}_1 \\
&= 0
\end{aligned}$$

(iv) *Antisymmetry:* $B_{jk} = -B_{kj}$ implies

$$\begin{aligned}
\{f,g\}_B &= \{f,g\} - \sum_{j,k=1}^{3} B_{jk}\, \partial_{p_j} f\, \partial_{p_k} g \\
&\overset{[1]}{=} -\{g,f\} + \sum_{j,k=1}^{3} B_{kj}\, \partial_{p_j} f\, \partial_{p_k} g \\
&\overset{[1]}{=} -\{g,f\}_B
\end{aligned}$$

Derivation property:

$$\begin{aligned}
\big\{f\,g, h\big\}_B &= \big\{f\,g, h\big\} - \sum_{j,k=1}^{3} B_{jk}\, \partial_{p_j}(f\,g)\, \partial_{p_k} h \\
&\overset{[1]}{=} f\,\{g,h\} + g\,\{f,h\} - \sum_{j,k=1}^{3} B_{jk}\, \big(\partial_{p_j} f\, g + f\, \partial_{p_j} g\big)\, \partial_{p_k} h \\
&\overset{[1]}{=} f\,\{g,h\}_B + g\,\{f,h\}_B
\end{aligned}$$