Operators

Homework Problems

12. Projections

Consider the multiplication operator $P = p(\hat{x})$ on $L^2(\mathbb{R}^d)$ associated to the function

$$p(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

(i) Find 2 eigenfunctions.

(ii) Compute $\sigma(P)$.

(iii) Determine the nature of the spectrum, i. e. determine $\sigma_{\mathrm{p}}(P)$, $\sigma_{\mathrm{cont}}(P)$, and $\sigma_{\mathrm{r}}(P)$.

(iv) Prove that P is an orthogonal projection.

13. The discrete Laplacian

Consider the Hilbert space $\ell^2(\mathbb{Z})$ with the usual scalar product $\langle \cdot , \cdot \rangle_{\ell^2(\mathbb{Z})}$. Define the shift operator

$$\mathfrak{s} : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}), \; (\mathfrak{s}\psi)(n) := \psi(n-1)$$

as well as the shift by $a \in \mathbb{Z}$ lattice units, $\mathfrak{s}_a := \mathfrak{s}^a$. Consider the *discrete Laplacian*

$$\Delta : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}), \; (\Delta\psi)(n) := \psi(n+1) + \psi(n-1) - 2\psi(n).$$

(i) Compute $\mathfrak{s}_a^*$ and prove that $\mathfrak{s}_a$ is unitary.

(ii) Show that Δ is a bounded operator on $\ell^2(\mathbb{Z})$.

(iii) Show that $\mathfrak{s}_a$ and Δ commute, i. e. $[\mathfrak{s}_a, \Delta] := \mathfrak{s}_a\Delta - \Delta\mathfrak{s}_a = 0$.

(iv) Compute Δ^*.

(v) Determine E_k so that

$$\psi_k(n) := \mathrm{e}^{+\mathrm{i}nk}, \qquad n \in \mathbb{Z}, k \in [-\pi, +\pi],$$

is an eigenvalue to the discrete Laplacian,

$$(\Delta\psi_k)(n) = E_k\psi_k(n).$$

Is ψ_k an element of $\ell^2(\mathbb{Z})$?

Remark: The Hilbert space $\ell^2(\mathbb{Z})$ is often used in solid state physics where the shift operator $(\mathfrak{s}\widehat{\psi})(n) := \widehat{\psi}(n-1)$ is interpreted as translating the particle by one lattice unit.

14. Position and momentum representation

Consider $\ell^2(\mathbb{Z})$ with the usual scalar $\langle \cdot\,, \cdot \rangle_{\ell^2(\mathbb{Z})}$ product and $L^2([0, 2\pi])$ endowed with the scalar product

$$\langle \widehat{\varphi}, \widehat{\psi} \rangle_{L^2([0,2\pi])} := \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}k \, \overline{\widehat{\varphi}(k)} \, \widehat{\psi}(k).$$

Define the Fourier transform

$$\begin{aligned} \mathcal{F} : L^2([0, 2\pi]) &\longrightarrow \ell^2(\mathbb{Z}), \\ \psi(n) = \big(\mathcal{F}\widehat{\psi}\big)(n) &:= \big\langle \mathrm{e}^{+\mathrm{i}nk}, \widehat{\psi} \big\rangle_{L^2([0,2\pi])} \end{aligned}$$

and its inverse

$$\ell^2(\mathbb{Z}) \ni \psi \mapsto \big(\mathcal{F}^{-1}\psi\big)(k) = \sum_{n\in\mathbb{Z}} \psi(n) \, \mathrm{e}^{+\mathrm{i}nk}.$$

You may use without proof that $\mathcal{F}$ is unitary.

(i) For the shift operator $(\mathfrak{s}\widehat{\psi})(n) := \widehat{\psi}(n-1)$, compute $\mathcal{F}^{-1} \, \mathfrak{s} \, \mathcal{F}$.

(ii) For the discrete Laplacian from problem 13, compute the momentum representation $\mathcal{F}^{-1} \, \Delta \, \mathcal{F}$.

(iii) What is the connection between ψ_k from problem 13 (v) in the position representation and $\mathcal{F}^{-1} \, \Delta \, \mathcal{F}$ in the momentum representation? Heuristically, what is the inverse Fourier transform of ψ_k?

(iv) Is $\Delta \geq 0$? Justify your answer.

15. Rank-1 operators

Suppose $\varphi, \psi \neq 0$ are elements of a Hilbert space $\mathcal{H}$, and define the rank-1 operator $T = |\varphi\rangle\langle\psi|$ via

$$T\phi = \langle \psi, \phi \rangle \, \varphi.$$

(i) Find all eigenvectors and eigenvalues of T.

(ii) Compute $\sigma(T)$.

(iii) Determine the nature of the spectrum, i. e. determine $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{cont}}(T)$ and $\sigma_{\mathrm{r}}(T)$.

Hand in home work on: Friday, 10 October 2014, before class