UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 4
(2013.10.03)

Classical Mechanics

Homework Problems

11. The Gaußian integral (4 points)

Show that

$$\int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

12. Angular momentum as generator of rotations (11 points)

Consider the angular momentum observable $L(q,p) = \big(L_1(q,p), L_2(q,p), L_3(q,p)\big) := q \times p$. Show that L generates rotations:

(i) Solve

$$\frac{\mathrm{d}}{\mathrm{d}\omega} q(\omega) = \big\{L_1, q(\omega)\big\}, \qquad\qquad q(0) = q_0 \in \mathbb{R}^3,$$

explicitly.

(ii) Give the solution to

$$\frac{\mathrm{d}}{\mathrm{d}\omega} p(\omega) = \big\{L_1, p(\omega)\big\}, \qquad\qquad p(0) = p_0 \in \mathbb{R}^3,$$

explicitly. (You need not calculate the same thing twice.)

(iii) Give the flow Ψ to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \{L_1, q\} \\ \{L_1, p\} \end{pmatrix}.$$

Does Ψ exist for all $\omega \in \mathbb{R}$?

13. Averages with respect to states & the spectrum of observables (20 points)

(i) Show that the Gaußian measure

$$\mu_{a,b}(A) := \frac{1}{\pi ab} \int_A \mathrm{d}q \, \mathrm{d}p \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}}, \qquad\qquad A \subset \mathbb{R}^2 \text{ Borel set},$$

localized around a point in phase space $(q_0, p_0) \in \mathbb{R}^2$ of widths $a, b > 0$ is a classical state in the sense of Definition 3.1.1. (You need not prove that $\mu_{a,b}$ is a Borel measure.)

(ii) Compute the energy average

$$\mathbb{E}_{\mu_{a,b}}(H) = \frac{1}{\pi ab} \int_{\mathbb{R}^2} \mathrm{d}q \, \mathrm{d}p \, \mathrm{e}^{-\frac{(q-q_0)^2}{a^2}} \, \mathrm{e}^{-\frac{(p-p_0)^2}{b^2}} \, H(q,p)$$

for the *one-dimensional* harmonic oscillator Hamiltonian $H(q,p) = \frac{1}{2}(p^2 + q^2)$ with respect to the Gaußian state $\mu_{a,b}$.

(iii) Show that $\lim_{a,b \to 0} \mathbb{E}_{\mu_{a,b}}(H) = H(q_0, p_0)$.

(iv) Now consider the case where phase space is $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. Show that the each of the three components of angular momentum $L(q,p) = q \times p$ are constants of motion for the *three-dimensional* harmonic oscillator dynamics generated by $H_{\mathbb{R}^3}(q,p) := \sum_{j=1}^3 H(q_j, p_j)$.

(v) Give the spectrum for the observables q_1, p_1, L_1 and H.

14. Magnetic classical systems (17 points)

Consider the equations of motion of a non-relativistic particle in three dimensions which is subjected to an electromagnetic field where $\mathbf{E} = -\nabla_q V$ is the electric field and $\mathbf{B} = \big(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\big)$. In other words, we are considering the Hamilton function H and the magnetic version of Hamilton's equations of motion

$$\begin{pmatrix} B & -\mathrm{id}_{\mathbb{R}^3} \\ +\mathrm{id}_{\mathbb{R}^3} & 0 \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \nabla_q H \\ \nabla_p H \end{pmatrix} \tag{1}$$

where the magnetic field matrix

$$B(q) = \begin{pmatrix} 0 & +\mathbf{B}_3 & -\mathbf{B}_2 \\ -\mathbf{B}_3 & 0 & +\mathbf{B}_1 \\ +\mathbf{B}_2 & -\mathbf{B}_1 & 0 \end{pmatrix}$$

is defined in terms of the components of $\mathbf{B}$. We denote the corresponding Hamiltonian flow with Φ.

Moreover, we define the *magnetic* Poisson bracket

$$\{f, g\}_B := \sum_{j=1}^{3} \big(\partial_{p_j} f \, \partial_{q_j} g - \partial_{q_j} f \, \partial_{p_j} g\big) - \sum_{j,k=1}^{3} B_{jk} \, \partial_{p_j} f \, \partial_{p_k} g.$$

(i) Show that $\{\cdot\,,\cdot\}_B$ generates equations (1).

(Hint: Consider the equations of motion generated by H for q and p in the Heisenberg picture.)

(ii) Show that $\mathbf{B}$ is source-free, i. e. $\nabla_q \cdot \mathbf{B} = 0$.

(Hint: Rewrite the magnetic field $\mathbf{B} = \nabla_q \times \mathbf{A}$ as the curl of a vector potential $\mathbf{A}$.)

(iii) Show that $\{f, g\}_B$ is antisymmetric and has the derivation property (cf. Proposition 3.3.4).

(iv) Show that energy is a constant of motion by computing the time-derivative of $H(t) := H \circ \Phi_t$.

Hand in home work on: Thursday, 10 October 2013, before class