Unitary Evolution Group, Resolvents & Symmetric Operators

Homework Problems

16. Translation semigroup on $L^2\big([0,+\infty)\big)$ (24 points)

(i) Show that for $t \geq 0$, the translation operator

$$\big(T_t\psi\big)(x) := \begin{cases} 0 & t \in [0,t) \\ \psi(x-t) & x \in [t,+\infty) \end{cases}$$

preserves angles on $L^2\big([0,+\infty)\big)$, i. e. $\big\langle T_t\psi, T_t\varphi\big\rangle = \langle\psi,\varphi\rangle$.

(ii) Compute the adjoint of T_t.

(iii) Show that $\{T_t\}_{t\in[0,+\infty)}$ forms a *semigroup*, i. e. $T_{t_1}\,T_{t_2} = T_{t_1+t_2}$ holds for all $t_1, t_2 \in [0,+\infty)$ and $T_0 = \mathrm{id}$.

(iv) Find the generator of $\{T_t\}_{t\in[0,+\infty)}$.
(A formal computation ignoring domain questions suffices.)

(v) Is the generator of $\{T_t\}_{t\in[0,+\infty)}$ symmetric on $\mathcal{C}^\infty_{\mathrm{c}}\big([0,+\infty)\big)$? Justify your answer.

(vi) Find a domain such that the generator of $\{T_t\}_{t\in[0,+\infty)}$ is symmetric.

(vii) Can $\{T_t\}_{t\in[0,+\infty)}$ be extended to a unitary evolution group? Justify your answer.

Solution:

(i) For $\psi,\varphi \in L^2\big([0,+\infty)\big)$ we compute

$$\begin{aligned}\big\langle T_t\psi, T_t\varphi\big\rangle &\overset{[1]}{=} \int_0^{+\infty}\mathrm{d}x\,\overline{\big(T_t\psi\big)(x)}\,\big(T_t\varphi\big)(x) \overset{[1]}{=} \int_0^t \mathrm{d}x\,0 + \int_t^{+\infty}\mathrm{d}x\,\overline{\psi(x-t)}\,\varphi(x-t) \\ &\overset{[1]}{=} \int_0^{+\infty}\mathrm{d}x\,\overline{\psi(x)}\,\varphi(x) \overset{[1]}{=} \langle\psi,\varphi\rangle.\end{aligned}$$

(ii) A quick computation reveals that the adjoint operator is $(T_y^*\psi)(x) = \psi(x+y)$ [1]:

$$\begin{aligned}\big\langle T_t\psi, \varphi\big\rangle &\overset{[1]}{=} \int_0^{+\infty}\mathrm{d}x\,\overline{\big(T_t\psi\big)(x)}\,\varphi(x) \\ &\overset{[1]}{=} \int_0^t \mathrm{d}x\,0 + \int_t^{+\infty}\mathrm{d}x\,\overline{\psi(x-t)}\,\varphi(x) \\ &\overset{[1]}{=} \int_0^{+\infty}\mathrm{d}x\,\overline{\psi(x)}\,\varphi(x+t) \overset{[1]}{=} \big\langle\psi, T_t^*\varphi\big\rangle\end{aligned}$$

(iii) That $T_0 = \mathrm{id}$ is clear from the definition. Pick $t_1, t_2 \in [0, +\infty)$ and $\psi \in L^2\big([0,+\infty)\big)$. Then we get from the definition

$$\big(T_{t_1}\, T_{t_2}\psi\big)(x) \overset{[1]}{=} \begin{cases} 0 & x \in [0, t_1) \\ \big(T_{t_2}\psi\big)(x - t_1) & x \in [t_1, +\infty) \end{cases}$$
$$\overset{[1]}{=} \begin{cases} 0 & x \in [0, t_1) \\ 0 & x \in [t_1, t_2) \\ \psi(x - t_1 - t_2) & x \in [t_1, +\infty) \end{cases}$$
$$\overset{[1]}{=} \big(T_{t_1+t_2}\psi\big)(x).$$

(iv) For the purpose of the computation we may assume $x > 0$. The formally, we obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\big(T_t\psi\big)(x)\bigg|_{t=0} \overset{[1]}{=} \mathrm{i}\partial_t\big(\psi(x-t)\big)\big|_{t=0} \overset{[1]}{=} -\mathrm{i}\partial_x\psi(x-t)\big|_{t=0}$$
$$\overset{[1]}{=} -\mathrm{i}\partial_x\psi(x).$$

(v) The generator $-\mathrm{i}\partial_x$ is not symmetric on $\mathcal{C}^\infty_{\mathrm{c}}\big([0,+\infty)\big)$, because the boundary terms do not vanish [1]:

$$\big\langle -\mathrm{i}\partial_x\psi, \varphi\big\rangle \overset{[1]}{=} +\mathrm{i}\int_0^{+\infty} \mathrm{d}x\, \overline{\partial_x\psi(x)}\,\varphi(x)$$
$$\overset{[1]}{=} +\mathrm{i}\Big[\overline{\psi(x)}\,\varphi(x)\Big]_0^{+\infty} - \mathrm{i}\int_0^{+\infty}\mathrm{d}x\,\overline{\psi(x)}\,\partial_x\varphi(x)$$
$$\overset{[1]}{=} -\mathrm{i}\,\overline{\psi(0)}\,\varphi(0) + \big\langle \psi, -\mathrm{i}\partial_x\varphi\big\rangle$$

(vi) We need to ensure $\psi(0) = 0$ [1], so $\mathcal{D} := \big\{\psi \in \mathcal{C}^\infty_{\mathrm{c}}\big([0,+\infty)\big) \;\mid\; \psi(0) = 0\big\}$ will do [1].

(vii) No, the adjoint $(T_t^*\psi) = \psi(x+t)$ is not norm-preserving [1], and hence, not unitary [1]. (For instance, if we fix $t > 0$ and pick $0 \neq \varphi \in L^2\big([0,+\infty)\big)$ so that $\varphi(x)$ is 0 almost everywhere as long as $x \geq t$, then $T_t^*\varphi = 0$.) Thus, $\{T_t\}_{t\in[0,+\infty)}$ cannot be extended to a unitary evolution group [1].

17. Convergence of operators (20 points)

Consider the following sequences $\{T_n\}_{n\in\mathbb{N}}$ of operators on the Hilbert space

$$\ell^2(\mathbb{N}) = \Big\{ a \equiv (a_n)_{n\in\mathbb{N}} \;\Big|\; \textstyle\sum_{n=1}^{\infty} |a_n|^2 < \infty \Big\}$$

and investigate whether they converge in norm, strongly or weakly:

(i) $T_n(a) := \big(\frac{1}{n} a_1, \frac{1}{n} a_2, \ldots\big)$

(ii) $T_n(a) := \big(\underbrace{0, \ldots, 0}_{n \text{ places}}, a_{n+1}, a_{n+2}, \ldots\big)$

(iii) $T_n(a) := \big(\underbrace{0, \ldots, 0}_{n \text{ places}}, a_1, a_2, \ldots\big)$

Solution:

(i) The sequence T_n converges in norm/uniformly to $0 \in \mathcal{B}\big(\ell^2(\mathbb{N})\big)$ [1], because

$$\big\|T_n(a)\big\|_{\ell^2(\mathbb{N})} = \tfrac{1}{n} \, \|a\|_{\ell^2(\mathbb{N})} \xrightarrow{n\to\infty} 0, \qquad [1]$$

and thus $\big\|T_n\big\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = {}^1\!/_n$. The above equation also implies that T_n converges to 0 also strongly and weakly [2], because

$$\big|\big\langle a, T_n(b)\big\rangle_{\ell^2(\mathbb{N})}\big| \leq \big\|a\big\|_{\ell^2(\mathbb{N})} \, \big\|T_n(b)\big\|_{\ell^2(\mathbb{N})} \xrightarrow{n\to\infty} 0\,. \qquad [1]$$

(ii) For a fixed $a \in \ell^2(\mathbb{N})$, we have

$$\big\|T_n(a)\big\|_{\ell^2(\mathbb{N})} \overset{[1]}{=} \sum_{j=1}^{\infty} \big|\big(T_n(a)\big)_j\big|^2 = \sum_{j=n+1}^{\infty} |a_j| \xrightarrow{n\to\infty} 0\,, \qquad [1]$$

and thus T_n converges strongly (and weakly) to $0 \in \mathcal{B}\big(\ell^2(\mathbb{N})\big)$ [2]. However, if $e_n := (\delta_{jn})_{j\in\mathbb{N}} = \big(0, \ldots, 0, 1, 0, \ldots\big)$, we see that

$$\big\|T_n e_{n+1}\big\|_{\ell^2(\mathbb{N})} \overset{[1]}{=} 1\,,$$

and thus T_n does not converge to 0 in norm [1], because $\|T_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \geq 1$ [1].

(iii) T_n converges weakly to 0 [1]:

$$\begin{aligned}
\Big|\big\langle a, T_n(b)\big\rangle_{\ell^2(\mathbb{N})}\Big| &\overset{[1]}{=} \left|\sum_{j=1}^{\infty} \overline{b_j}\,\big(T_n(a)\big)_j\right| \overset{[1]}{=} \left|\sum_{j=n+1}^{\infty} \overline{b_j}\, a_{j-n}\right| \\
&\overset{[1]}{\leq} \left(\sum_{j=n+1}^{\infty} |b_j|^2\right)^{1/2} \left(\sum_{j=n+1}^{\infty} \big|a_{j-n}\big|^2\right)^{1/2} \overset{[1]}{=} \left(\sum_{j=n+1}^{\infty} |b_j|^2\right)^{1/2} \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{1/2} \\
&\xrightarrow{n\to\infty} 0. \qquad [1]
\end{aligned}$$

However, it does not converge strongly or in norm, because

$$\big\|T_n(a)\big\|^2_{\ell^2(\mathbb{N})} \overset{[1]}{=} \sum_{j=n+1}^{\infty} \big|a_{j-n}\big|^2 \overset{[1]}{=} \sum_{j=1}^{\infty} |a_j|^2 \overset{[1]}{=} \|a\|_{\ell^2(\mathbb{N})}\,.$$

18. The resolvent (27 points)

Let $S, T \in \mathcal{B}(\mathcal{X})$ be operators on a Banach space $\mathcal{X}$ with resolvent sets $\rho(S)$ and $\rho(T)$. On these sets, the *resolvents* $(T-z)^{-1}$ and $(S-z)^{-1}$ exist as bounded operators.

(i) Prove the first resolvent identity, i. e. that for any $z, z' \in \rho(T)$ we have

$$(T-z)^{-1} - (T-z')^{-1} = (z-z')\,(T-z)^{-1}\,(T-z')^{-1}.$$

(ii) Prove the second resolvent identity, i. e. that for any $z \in \rho(T) \cap \rho(S)$ we have

$$(T-z)^{-1} - (S-z)^{-1} = (T-z)^{-1}\,(T-S)\,(S-z)^{-1}.$$

(iii) Prove that if $\|T\| < 1$, then the geometric series $\sum_{n=0}^{\infty} T^n$ exists in $\mathcal{B}(\mathcal{X})$ and equals $(\mathrm{id}-T)^{-1}$.

(iv) Show that the resolvent set $\rho(T) \subseteq \mathbb{C}$ is open and the resolvent is $z \mapsto (T-z)^{-1}$ is analytic on $\rho(T)$, meaning locally there exists a power series expansion of $(T-z)^{-1}$ which converges in operator norm.

(v) Show that the spectrum $\sigma(T) \subseteq \mathbb{C}$ is closed.

Solution:

(i) If we multiply the left-hand side with $T-z$ from the left and $T-z'$ from the right [1], we get

$$(T-z)\big((T-z)^{-1} - (T-z')^{-1}\big)(T-z') \overset{[1]}{=} (T-z') - (T-z) \overset{[1]}{=} z - z'.$$

Evidently, this is equivalent to the first resolvent identity.

(ii) We again multiply with $T-z$ from the left and $S-z$ from the right [1], and obtain

$$(T-z)\big((T-z)^{-1} - (S-z)^{-1}\big)(S-z) \overset{[1]}{=} (S-z) - (T-z) \overset{[1]}{=} S - T.$$

(iii) First of all, the sequence of partial sums $S_N := \sum_{n=0}^{N} T^n$ is Cauchy [1], because for $N > M$

$$\big\|S_N - S_M\big\| = \left\| \sum_{n=M+1}^{N} T^n \right\| \overset{[1]}{\leq} \sum_{n=M+1}^{N} \|T\|^n$$

and the right-hand side is finite since $\|T\| < 1$ [1], and goes to 0 as $N, M \to \infty$ [1]. By the completeness of $\mathcal{B}(\mathcal{X})$ (Proposition 4.1.4) [1], the sequence of partial sums S_N converges to some $S \in \mathcal{B}(\mathcal{X})$ [1]. To show $S = (\mathrm{id}-T)^{-1}$, we compute

$$\begin{aligned}(\mathrm{id}-T)\,S &\overset{[1]}{=} \lim_{N\to\infty} (\mathrm{id}-T) \sum_{n=0}^{N} T^n \\ &\overset{[1]}{=} \lim_{N\to\infty} \big(\mathrm{id} - T^{N+1}\big) \overset{[1]}{=} \mathrm{id}.\end{aligned}$$

(iv) Let $z_0 \in \rho(T)$. We first show that there exists an $\varepsilon > 0$ such that $\big\{z \in \mathbb{C} \mid |z-z_0| < \varepsilon\big\} \subseteq \rho(T)$. Pick $0 < \varepsilon < \big\|(T-z_0)^{-1}\big\|^{-1}$ [1]. Then

$$(T-z) \overset{[1]}{=} (T-z_0)\Big(\mathrm{id} - (z-z_0)\,(T-z_0)^{-1}\Big)$$

is invertible [1], because $(T - z_0)^{-1}$ exists and in view of

$$\left\| (z - z_0)\,(T - z_0)^{-1} \right\| < \varepsilon \left\| (T - z_0)^{-1} \right\| \overset{[1]}{<} 1$$

(iii) tells us that also the second term in the product is invertible [1]. That means $\rho(T)$ is open [1].

(iii) also gives us an explicit expression for the power series of the resolvent in this neighborhood,

$$(T - z)^{-1} \overset{[1]}{=} \Bigl(\mathrm{id} - (z - z_0)\,(T - z_0)^{-1} \Bigr)^{-1} (T - z_0)^{-1} \overset{[1]}{=} \sum_{n=0}^{\infty} (z - z_0)^n \,(T - z_0)^{-(n+1)}.$$

Hence, $(T - z)^{-1}$ is analytic in a vicinity of z_0 [1], and since the point was chosen arbitrarily, the resolvent is in fact analytic on all of $\rho(T)$ [1].

(v) Since $\rho(T)$ is open by (iv) [1], its complement in $\mathbb{C}$, the spectrum $\sigma(T)$, is closed [1].

19. Symmetric operators (17 points)

Let $H = \frac{1}{2m}(-\mathrm{i}\nabla_x)^2 + V$ be a Hamilton operator with potential $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$.
Define the smooth functions with compact support as

$$\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3) := \bigl\{\varphi : \mathbb{R}^3 \longrightarrow \mathbb{C} \mid \varphi \in \mathcal{C}^{\infty}(\mathbb{R}^3),\ \operatorname{supp}\varphi \text{ compact}\bigr\}.$$

(i) Prove $\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$.

(ii) Show that H is *symmetric* on $\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3)$, i. e. that

$$\bigl\langle \varphi, H\psi \bigr\rangle = \bigl\langle H\varphi, \psi \bigr\rangle$$

holds for all $\varphi, \psi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3)$.

Solution:

(i) Every smooth function with compact support is square-integrable: let $\varphi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3)$, then there exists a compact subset $K \subset \mathbb{R}^3$, so that

$$\operatorname{supp}\varphi = \overline{\bigl\{x \in \mathbb{R}^3 \mid \varphi(x) \neq 0\bigr\}} \subseteq K\,. \qquad [1]$$

Since φ is also continuous, we can estimate the supremum from above by

$$\lVert\varphi\rVert_{\infty} = \sup_{x \in \mathbb{R}^3} \lvert\varphi(x)\rvert = \sup_{x \in K} \lvert\varphi(x)\rvert < \infty\,. \qquad [1]$$

Hence, we obtain

$$\lVert\varphi\rVert^2 = \int_{\mathbb{R}^3} \mathrm{d}x\ \lvert\varphi(x)\rvert^2 \overset{[1]}{=} \int_K \mathrm{d}x\ \lvert\varphi(x)\rvert^2 \overset{[1]}{\leq} \lvert K\rvert\ \bigl(\sup_{x \in K} \lvert\varphi(x)\rvert\bigr)^2 < \infty.$$

(ii) We will treat kinetic and potential energy separately: clearly, derivatives map $\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3)$ into itself, and thus $(-\mathrm{i}\nabla_x)^2\varphi \in L^2(\mathbb{R}^3)$ [1]. Fix $\varphi, \psi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3)$. Then there exists a compact set $K \subset \mathbb{R}^3$ whose *interior* contains $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ [1]. Then we compute using repeated partial integration

$$\begin{aligned}
\bigl\langle \varphi, \tfrac{1}{2m}(-\mathrm{i}\nabla_x)^2\psi \bigr\rangle &= \sum_{j=1}^{3} \frac{1}{2m} \bigl\langle \varphi, (-\mathrm{i}\partial_{x_j})^2\psi \bigr\rangle \overset{[1]}{=} \sum_{j=1}^{3} \frac{1}{2m} \int_{\mathbb{R}^3} \mathrm{d}x\, \overline{\varphi(x)}\, \bigl((-\mathrm{i}\partial_{x_j})^2\psi\bigr)(x) \\
&\overset{[1]}{=} \sum_{j=1}^{3} \frac{1}{2m} \int_K \mathrm{d}x\, \overline{\varphi(x)}\, \bigl((-\mathrm{i}\partial_{x_j})^2\psi\bigr)(x) \\
&\overset{[1]}{=} \sum_{j=1}^{3} \frac{1}{2m} \int_{\partial K} \mathrm{d}S(x)\, \overline{\varphi(x)}\, \bigl((-\mathrm{i})^2\partial_{x_j}\psi\bigr)(x) + \\
&\qquad - \sum_{j=1}^{3} \frac{1}{2m} \int_K \mathrm{d}x\, \overline{\partial_{x_j}\varphi(x)}\, \bigl((-\mathrm{i})^2\partial_{x_j}\psi\bigr)(x) \\
&\overset{[1]}{=} 0 - \sum_{j=1}^{3} \frac{1}{2m} (-\mathrm{i})^2 \int_{\partial K} \mathrm{d}S(x)\, \overline{\partial_{x_j}\varphi(x)}\, \psi(x) + \\
&\qquad + \sum_{j=1}^{3} \frac{1}{2m} (-\mathrm{i})^2 \int_K \mathrm{d}x\, \overline{\partial_{x_j}^2\varphi(x)}\, \psi(x)
\end{aligned}$$

$$\overset{[1]}{=} \sum_{j=1}^{3} \frac{1}{2m} \int_K \mathrm{d}x \, \overline{\big((-\mathrm{i}\partial_{x_j})^2 \varphi\big)(x)} \, \psi(x)$$

$$\overset{[1]}{=} \big\langle \tfrac{1}{2m} (-\mathrm{i}\nabla_x)^2 \varphi, \psi \big\rangle$$

Here, $\mathrm{d}S(x)$ is the surface measure on ∂K. The boundary terms vanish, because φ and ψ as well as their derivatives vanish on ∂K.

Now to the potential energy: since V is continuous, it is bounded on compact subsets. Choose any $\varphi, \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$. Then $V\varphi \in L^2(\mathbb{R}^3)$ [1] and hence,

$$\big\langle \varphi, V\psi \big\rangle \overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\varphi(x)} \, \big(V\psi\big)(x) \overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\varphi(x)} \, V(x) \, \psi(x)$$

$$\overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\big(V\varphi\big)(x)} \, \psi(x) \overset{[1]}{=} \big\langle V\varphi, \psi \big\rangle$$

holds.