UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

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Solutions 5
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Banach Spaces, the Convolution & Solutions to the Heat Equation

Homework Problems

15. Direct sum of Banach spaces (5 points)

Assume $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$. Show that the *direct sum* $\mathcal{X} \oplus \mathcal{Y}$ defined as the product space $\mathcal{X} \times \mathcal{Y}$ equipped with

$$\|(x, y)\|_{\mathcal{X}\oplus\mathcal{Y}} := \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$$

is a Banach space.

Solution:

First of all, we need to verify that $\|\cdot\|_{\mathcal{X}\oplus\mathcal{Y}}$ is a norm. Let us start with definiteness: if

$$\|(x, y)\|_{\mathcal{X}\oplus\mathcal{Y}} = \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}} = 0\,,$$

then $\|x\|_{\mathcal{X}} = 0$ and $\|y\|_{\mathcal{Y}} = 0$ which implies $x = 0$ and $y = 0$, i. e. $(x, y) = 0 \in \mathcal{X} \times \mathcal{Y}$ [1].

If $\lambda \in \mathbb{C}$ is a scalar, then

$$\begin{aligned}
\|\lambda(x, y)\|_{\mathcal{X}\oplus\mathcal{Y}} &= \big\|\big(\lambda x, \lambda y\big)\big\|_{\mathcal{X}\oplus\mathcal{Y}} \\
&= \|\lambda x\|_{\mathcal{X}} + \|\lambda y\|_{\mathcal{Y}} \\
&= \lambda\, \|x\|_{\mathcal{X}} + \lambda\, \|y\|_{\mathcal{Y}} \\
&= \lambda\, \|(x, y)\|_{\mathcal{X}\oplus\mathcal{Y}}\,. && [1]
\end{aligned}$$

Lastly, the triangle equality for $\|\cdot\|_{\mathcal{X}\oplus\mathcal{Y}}$ follows from the triangle inequalities for $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$:

$$\begin{aligned}
\big\|(x, y) + (x', y')\big\|_{\mathcal{X}\oplus\mathcal{Y}} &= \big\|\big(x + x', y + y'\big)\big\|_{\mathcal{X}\oplus\mathcal{Y}} \\
&= \big\|x + x'\big\|_{\mathcal{X}} + \big\|y + y'\big\|_{\mathcal{Y}} \\
&\leq \|x\|_{\mathcal{X}} + \|x'\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}} + \|y'\|_{\mathcal{Y}} \\
&= \|(x, y)\|_{\mathcal{X}\oplus\mathcal{Y}} + \big\|(x', y')\big\|_{\mathcal{X}\oplus\mathcal{Y}} && [1]
\end{aligned}$$

To verify that $\mathcal{X} \oplus \mathcal{Y}$ is complete, let $z_n = (x_n, y_n)$ be a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{X}\oplus\mathcal{Y}}$. Then also (x_n) and (y_n) are Cauchy sequences in $\mathcal{X}$ and $\mathcal{Y}$, respectively, which converge to $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ [1]. Hence, (x_n, y_n) converges to (x_0, y_0), and $\mathcal{X} \oplus \mathcal{Y}$ is complete, hence, a Banach space [1].

16. The convolution on $L^1(\mathbb{R}^n)$ (12 points)

Define the convolution of f and g to be

$$f * g(x) := \int_{\mathbb{R}^n} \mathrm{d}y\, f(x-y)\, g(y).$$

Prove the following statements:

(i) $f, g \in L^1(\mathbb{R}^n) \Rightarrow f * g \in L^1(\mathbb{R}^n)$

(ii) $f * g = g * f$

(iii) $(f * g) * h = f * (g * h)$

Solution:

(i) To show that a function is in $L^1(\mathbb{R}^n)$, we need to show that its L^1-norm is finite [1]:

$$\begin{aligned}
\|f * g\|_1 &\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}x\, \big|(f * g)(x)\big| \\
&= \int_{\mathbb{R}^n} \mathrm{d}x\, \left| \int_{\mathbb{R}^n} \mathrm{d}y\, f(x-y)\, g(y) \right| \\
&\overset{[1]}{\leq} \int_{\mathbb{R}^n} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y\, \big|f(x-y)\, g(y)\big| \\
&\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}x'\, \big|f(x')\big| \int_{\mathbb{R}^n} \mathrm{d}y\, \big|g(y)\big| \\
&\overset{[1]}{=} \|f\|_1\, \|g\|_1 < \infty
\end{aligned}$$

Hence, $f * g$ is integrable [1].

(ii) This follows from a simple change of variables:

$$\begin{aligned}
f * g(x) &\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}y\, f(x-y)\, g(y) \\
&\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}y'\, f(y')\, g(x-y') = g * f(x)
\end{aligned}$$

(iii) This follows from plugging in the definition, a change of variables and the fact that under these circumstances, we may change the order of integration:

$$\begin{aligned}
\big((f * g) * h\big)(x) &\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}z\, (f * g)(x-z)\, h(z) \\
&\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}z \int_{\mathbb{R}^n} \mathrm{d}y\, f(x-y-z)\, g(y)\, h(z) \\
&\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}z \int_{\mathbb{R}^n} \mathrm{d}y'\, f(x-y')\, g(y'-z)\, h(z) \\
&\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}y' f(x-y')\, (g * h)(y') = \big(f * (g * h)\big)(x)
\end{aligned}$$

17. Exchanging limits and integration (25 points)

(i) Let $g \in \mathcal{C}^1\big(\mathbb{R}, L^1(\mathbb{R}^n)\big)$ a parameter-dependent function with values in $L^1(\mathbb{R}^n)$ so that there exists $h \in L^1(\mathbb{R}^n)$ with $\big|\partial_\lambda g(\lambda, x)\big| \leq h(x)$ for almost all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$. Show that differentiation with respect to λ and integration commute, i. e.

$$\frac{\partial}{\partial \lambda} \int_{\mathbb{R}^n} \mathrm{d}x \, g(\lambda, x) = \int_{\mathbb{R}^n} \mathrm{d}x \, \partial_\lambda g(\lambda, x) \, .$$

(ii) In addition to $f \in L^1(\mathbb{R}^n)$ assume $\lVert x_1 \, f \rVert_1 < \infty$. Then show that

$$\big(\mathrm{i}\partial_{\xi_1} \mathcal{F} f\big)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}x \cdot \xi} \, x_1 \, f(x)$$

holds.

(iii) In addition to $f, g \in L^1(\mathbb{R}^n)$ assume $\partial_{x_1} f, \partial_{x_1} g \in L^\infty(\mathbb{R}^n)$. Then show that

$$\partial_{x_1}(f * g) = \partial_{x_1} f * g = f * \partial_{x_1} g$$

holds.

Solution:

(i) The derivative of the integral can be written as differential quotient [1]: for $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \partial_\lambda \int_{\mathbb{R}^n} \mathrm{d}x \, g(\lambda, x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \bigg(\int_{\mathbb{R}^n} \mathrm{d}x \, g(\lambda + \varepsilon, x) - \int_{\mathbb{R}^n} \mathrm{d}x \, g(\lambda, x) \bigg) \\ &\overset{[1]}{=} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \mathrm{d}x \, \tfrac{1}{\varepsilon} \big(g(\lambda + \varepsilon, x) - g(\lambda, x) \big) \, . \end{aligned}$$

Using the Mean Value Theorem [1], we can estimate $\frac{1}{\varepsilon}\big(g(\lambda + \varepsilon, x) - g(\lambda, x)\big)$ by the function h. For almost all values of x, there exists $\lambda_0 \in \big(\lambda - \lvert\varepsilon\rvert \, , \lambda + \lvert\varepsilon\rvert\big)$ with

$$\Big| \tfrac{1}{\varepsilon} \big(g(\lambda + \varepsilon, x) - g(\lambda, x) \big) \Big| \overset{[1]}{=} \big| \partial_\lambda g(\lambda_0, x) \big| \overset{[1]}{\leq} \sup_{\lambda \in \mathbb{R}} \big| \partial_\lambda g(\lambda, x) \big| \overset{[1]}{\leq} h(x).$$

The upper bound h is *independent* of λ. Thus, the prerequisites of the Dominated Convergence Theorem are satisfied [1], and we can exchange limit and integration,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \mathrm{d}x \, \tfrac{1}{\varepsilon} \big(g(\lambda + \varepsilon, x) - g(\lambda, x) \big) &\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}x \, \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \big(g(\lambda + \varepsilon, x) - g(\lambda, x) \big) \\ &\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}x \, \partial_\lambda g(\lambda, x). \end{aligned}$$

(ii) This is only a special case of (i): here, the dominating function is $\big|x_1 \, f(x)\big|$ (which is integrable by assumption),

$$\Big| \mathrm{i}\partial_{\xi_1} \big(\mathrm{e}^{-\mathrm{i}x \cdot \xi} \, f(x) \big) \Big| \overset{[1]}{=} \big| x_1 \, \mathrm{e}^{-\mathrm{i}x \cdot \xi} \, f(x) \big| \overset{[1]}{\leq} \big| x_1 \, f(x) \big| \, ,$$

and the bound is independent of ξ. Thus, by (i) $\xi \mapsto (\mathcal{F}f)(\xi)$ is continuously differentiable, and we may interchange differentiation with respect to ξ and integration [1],

$$\begin{aligned}
\big(\mathrm{i}\partial_{\xi_1}\mathcal{F}f\big)(\xi) &\overset{[1]}{=} \mathrm{i}\partial_{\xi_1} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}x\cdot\xi} \, f(x) \\
&\overset{[1]}{=} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{d}x \, \mathrm{i}\partial_{\xi_1} \big(\mathrm{e}^{-\mathrm{i}x\cdot\xi} \, f(x)\big) \\
&\overset{[1]}{=} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}x\cdot\xi} \, x_1 \, f(x) \\
&= \big(\mathcal{F}(x_1 \, f)\big)(\xi) \, .
\end{aligned}$$

(iii) Pick an arbitrary $x \in \mathbb{R}$. Writing the derivative as a limit,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(f * g)(x) &\overset{[1]}{=} \lim_{\delta \to 0} \frac{(f * g)(x + \delta e_1) - (f * g)(x)}{\delta} \\
&\overset{[1]}{=} \lim_{\delta \to 0} \int_{\mathbb{R}} \mathrm{d}y \, \tfrac{1}{\delta}\big(f(x + \delta e_1 - y) - f(x - y)\big) \, g(y)
\end{aligned}$$

we see that we need to estimate $\frac{1}{\delta}\big(f(x + \delta e_1 - y) - f(x - y)\big)$ the integrand [1]. Assume for simplicity that $\delta > 0$. Then the mean value theorem states that there exits $x_0 \in (x, x + \delta)$ with

$$\begin{aligned}
\Big|\tfrac{1}{\delta}\big(f(x + \delta e_1 - y) - f(x - y)\big)\Big| &\overset{[1]}{=} \big|f'(x_0 - y)\big| \\
&\overset{[1]}{\leq} \sup_{x_0 \in \mathbb{R}^n} \big|f'(x_0 - y)\big| = C < \infty.
\end{aligned}$$

Hence, we can estimate the integrand by a constant times the integrable function g [1],

$$\big|\tfrac{1}{\delta}\big(f(x + \delta e_1 - y) - f(x - y)\big) \, g(y)\big| \overset{[1]}{\leq} \sup_{x \in \mathbb{R}^n} \big|f'(x)\big| \, \big|g(y)\big| = C \, \big|g(y)\big|.$$

Thus, Dominated Convergence applies and we can interchange the limit and differentiation [1],

$$\begin{aligned}
&\lim_{\delta \to 0} \int_{\mathbb{R}} \mathrm{d}y \, \tfrac{1}{\delta}\big(f(x + \delta e_1 - y) - f(x - y)\big) \, g(y) = \\
&\qquad \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}y \, \lim_{\delta \to 0} \tfrac{1}{\delta}\big(f(x + \delta e_1 - y) - f(x - y)\big) \, g(y) \\
&\qquad \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}y \, \partial_{x_1} f(x - y) \, g(y) = \big(\partial_{x_1} f * g\big)(x).
\end{aligned}$$

Using $f * g = g * f$, we may exchange the roles of f and g so that also $\partial_{x_1} f * g = f * \partial_{x_1} g$ [1].

18. Solving the heat equation using the convolution (22 points)

Consider the heat equation

$$\partial_t u(t,x) = D\partial_x^2 u(t,x), \qquad u(0,x) = f(x), \tag{1}$$

on $\mathbb{R}$ for $D > 0$. We will always assume $f \in L^1(\mathbb{R})$. For $t > 0$, define the function

$$G(t,x) := \frac{1}{\sqrt{4\pi Dt}} \, \mathrm{e}^{-\frac{x^2}{4Dt}} .$$

(i) Show that $u(t)$ is integrable, i. e. $u(t) \in L^1(\mathbb{R})$ holds for all $t \in \mathbb{R}$.

(ii) Show that the function $u(t) := G(t) * f$ solves (1).
(You may use $\lim_{t \searrow 0} G(t) * f = f$ for all $f \in L^1(\mathbb{R}^n)$ without proof.)

(iii) Show that for any $t > 0$, the solution $u(t)$ is smooth in x.

(iv) Show that for any $x \in \mathbb{R}$, $\lim_{t\to\infty} u(t,x) = 0$.

Solution:

(i) From problem 11 on sheet 4, we know that

$$\|G(t)\|_1 = \int_{\mathbb{R}} \mathrm{d}x \, \frac{1}{\sqrt{4\pi Dt}} \, \mathrm{e}^{-\frac{x^2}{4Dt}} = 1 \qquad [1]$$

for all $t > 0$ so that $G(t) \in L^1(\mathbb{R})$ [1]. Hence, by problem 16 (i) and $f \in L^1(\mathbb{R})$, the function $u(t) = G(t) * f \in L^1(\mathbb{R})$ is integrable as the convolution of two L^1-functions [1].

(ii) From problem 15 and 16, we know that

$$\partial_t u(t) \overset{[1]}{=} \big(\partial_t G(t)\big) * f$$

and

$$\partial_x^2 \big(G(t) * f\big) \overset{[1]}{=} \big(\partial_x^2 G(t)\big) * f$$

hold. Thus, all we need to compute are derivatives of $G(t)$. Let us start with the time derivative:

$$\begin{aligned}
\partial_t G(t,x) &= \frac{1}{\sqrt{4\pi D}} \left(-\frac{1}{2} t^{-3/2} + t^{-1/2} \left(-\frac{x^2}{4D} \right) \left(-t^{-2} \right) \right) \mathrm{e}^{-\frac{x^2}{4Dt}} \\
&\overset{[1]}{=} \frac{1}{\sqrt{4\pi Dt}} \left(-\frac{1}{2t} + \frac{x^2}{4Dt^2} \right) \mathrm{e}^{-\frac{x^2}{4Dt}}
\end{aligned}$$

Computing the second derivative with respect to x is straight-forward, too:

$$\begin{aligned}
\partial_x G(t,x) &\overset{[1]}{=} -\frac{1}{\sqrt{4\pi Dt}} \frac{x}{2Dt} \, \mathrm{e}^{-\frac{x^2}{4Dt}} \\
\partial_x^2 G(t,x) &= \frac{\partial}{\partial x} \left(-\frac{1}{\sqrt{4\pi Dt}} \frac{x}{2Dt} \, \mathrm{e}^{-\frac{x^2}{4Dt}} \right) \\
&= -\frac{1}{\sqrt{4\pi Dt}} \frac{1}{2Dt} \, \mathrm{e}^{-\frac{x^2}{4Dt}} + \frac{1}{\sqrt{4\pi Dt}} \left(\frac{x}{2Dt} \right)^2 \mathrm{e}^{-\frac{x^2}{4Dt}} \\
&\overset{[1]}{=} \frac{1}{\sqrt{4\pi Dt}} \left(-\frac{1}{2Dt} + \frac{x^2}{4D^2t^2} \right) \mathrm{e}^{-\frac{x^2}{4Dt}}
\end{aligned}$$

Hence, we compare the two and find that, up to a factor of D,

$$\partial_t G(t) = D\,\partial_x^2 G(t)\,.$$

Consequently, $u(t) = G(t) * f$ solves the heat equation,

$$\begin{aligned}\partial_t u(t,x) &= \partial_t\big(G(t) * f\big)(x) \overset{[1]}{=} \big(\partial_t G(t) * f\big)(x)\\ &\overset{[1]}{=} D\,\big(\partial_x^2 G(t) * f\big)(x)\\ &\overset{[1]}{=} D\,\partial_x^2 u(t,x).\end{aligned}$$

Moreover, it satisfies the initial condition as

$$u(0) = \lim_{t\searrow 0} u(t) = \lim_{t\searrow 0} G(t) * f = f. \qquad [1]$$

(iii) Seeing as $G(t)$ is smooth in x [1] and all derivatives are bounded, because they are of the form

$$\partial_x^n = p_n(x)\,\mathrm{e}^{-\frac{x^2}{4Dt}}$$

where p_n is a polynomial in x [1]. Hence, by problem 16, $\partial_x^n G(t) * f$ exists in $L^1(\mathbb{R})$ [1] and $u(t)$ is smooth in x for any $t > 0$ [1].

(iv) We note that the integrand of

$$u(t,x) = \big(G(t) * f\big)(x) = \int_{\mathbb{R}} \mathrm{d}y\, \frac{1}{\sqrt{4\pi Dt}}\,\mathrm{e}^{-\frac{(x-y)^2}{4Dt}}\, f(y)$$

is bounded by

$$\left|\frac{1}{\sqrt{4\pi Dt}}\,\mathrm{e}^{-\frac{(x-y)^2}{4Dt}}\, f(y)\right| \overset{[1]}{\leq} \frac{1}{\sqrt{4\pi Dt}}\,\big|f(y)\big| \overset{[1]}{\leq} \frac{1}{\sqrt{4\pi DT}}\,\big|f(y)\big|.$$

This estimate is independent of x and t as long as $t > T$. The right-hand side is integrable for all $t > 0$ and goes to 0 pointwise as $t \to \infty$ [1]. Thus, Dominated Convergence yields [1]

$$\begin{aligned}\lim_{t\to\infty} u(t,x) &= \lim_{t\to\infty} \int_{\mathbb{R}} \mathrm{d}y\, \frac{1}{\sqrt{4\pi Dt}}\,\mathrm{e}^{-\frac{(x-y)^2}{4Dt}}\, f(y)\\ &\overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}y\, \lim_{t\to\infty} \frac{1}{\sqrt{4\pi Dt}}\,\mathrm{e}^{-\frac{(x-y)^2}{4Dt}}\, f(y) \overset{[1]}{=} 0.\end{aligned}$$