UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Problem Sheet 5
(2014.10.10)

Unitary Evolution Group, Resolvents & Symmetric Operators

Homework Problems

16. Translation semigroup on $L^2\big([0,+\infty)\big)$

(i) Show that for $t \geq 0$, the translation operator

$$(T_t\psi)(x) := \begin{cases} 0 & t \in [0,t) \\ \psi(x-t) & x \in [t,+\infty) \end{cases}$$

preserves angles on $L^2\big([0,+\infty)\big)$, i. e. $\big\langle T_t\psi, T_t\varphi\big\rangle = \langle\psi,\varphi\rangle$.

(ii) Compute the adjoint of T_t.

(iii) Show that $\{T_t\}_{t\in[0,+\infty)}$ forms a *semigroup*, i. e. $T_{t_1}\,T_{t_2} = T_{t_1+t_2}$ holds for all $t_1, t_2 \in [0,+\infty)$ and $T_0 = \mathrm{id}$.

(iv) Find the generator of $\{T_t\}_{t\in[0,+\infty)}$.

(A formal computation ignoring domain questions suffices.)

(v) Is the generator of $\{T_t\}_{t\in[0,+\infty)}$ symmetric on $\mathcal{C}^\infty_{\mathrm{c}}\big([0,+\infty)\big)$? Justify your answer.

(vi) Find a domain such that the generator of $\{T_t\}_{t\in[0,+\infty)}$ is symmetric.

(vii) Can $\{T_t\}_{t\in[0,+\infty)}$ be extended to a unitary evolution group? Justify your answer.

17. Convergence of operators

Consider the following sequences $\{T_n\}_{n\in\mathbb{N}}$ of operators on the Hilbert space

$$\ell^2(\mathbb{N}) = \Big\{a \equiv (a_n)_{n\in\mathbb{N}} \;\Big|\; \textstyle\sum_{n=1}^\infty |a_n|^2 < \infty\Big\}$$

and investigate whether they converge in norm, strongly or weakly:

(i) $T_n(a) := \big(\tfrac{1}{n}a_1, \tfrac{1}{n}a_2, \ldots\big)$

(ii) $T_n(a) := \big(\underbrace{0,\ldots,0}_{n\text{ places}}, a_{n+1}, a_{n+2}, \ldots\big)$

(iii) $T_n(a) := \big(\underbrace{0,\ldots,0}_{n\text{ places}}, a_1, a_2, \ldots\big)$

18. The resolvent

Let $S, T \in \mathcal{B}(\mathcal{X})$ be operators on a Banach space $\mathcal{X}$ with resolvent sets $\rho(S)$ and $\rho(T)$. On these sets, the *resolvents* $(T-z)^{-1}$ and $(S-z)^{-1}$ exist as bounded operators.

(i) Prove the first resolvent identity, i. e. that for any $z, z' \in \rho(T)$ we have

$$(T-z)^{-1} - (T-z')^{-1} = (z-z')\,(T-z)^{-1}\,(T-z')^{-1}.$$

(ii) Prove the second resolvent identity, i. e. that for any $z \in \rho(T) \cap \rho(S)$ we have

$$(T-z)^{-1} - (S-z)^{-1} = (T-z)^{-1}\,(T-S)\,(S-z)^{-1}.$$

(iii) Prove that if $\|T\| < 1$, then the geometric series $\sum_{n=0}^{\infty} T^n$ exists in $\mathcal{B}(\mathcal{X})$ and equals $(\mathrm{id}-T)^{-1}$.

(iv) Show that the resolvent set $\rho(T) \subseteq \mathbb{C}$ is open and the resolvent is $z \mapsto (T-z)^{-1}$ is analytic on $\rho(T)$, meaning locally there exists a power series expansion of $(T-z)^{-1}$ which converges in operator norm.

(v) Show that the spectrum $\sigma(T) \subseteq \mathbb{C}$ is closed.

19. Symmetric operators

Let $H = \frac{1}{2m}(-\mathrm{i}\nabla_x)^2 + V$ be a Hamilton operator with potential $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$.

Define the smooth functions with compact support as

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3) := \bigl\{\varphi : \mathbb{R}^3 \longrightarrow \mathbb{C} \mid \varphi \in \mathcal{C}^\infty(\mathbb{R}^3),\ \operatorname{supp}\varphi \text{ compact}\bigr\}.$$

(i) Prove $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$.

(ii) Show that H is *symmetric* on $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$, i. e. that

$$\bigl\langle \varphi, H\psi \bigr\rangle = \bigl\langle H\varphi, \psi \bigr\rangle$$

holds for all $\varphi, \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$.

Hand in home work on: Friday, 17 October 2014, before class