UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 5
(2013.10.10)

Banach Spaces, the Convolution & Solutions to the Heat Equation

Homework Problems

15. Direct sum of Banach spaces (5 points)

Assume $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$. Show that the *direct sum* $\mathcal{X} \oplus \mathcal{Y}$ defined as the product space $\mathcal{X} \times \mathcal{Y}$ equipped with

$$\|(x,y)\|_{\mathcal{X}\oplus\mathcal{Y}} := \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$$

is a Banach space.

16. The convolution on $L^1(\mathbb{R}^n)$ (12 points)

Define the convolution of f and g to be

$$f * g(x) := \int_{\mathbb{R}^n} \mathrm{d}y \, f(x-y) \, g(y).$$

Prove the following statements:

(i) $f, g \in L^1(\mathbb{R}^n) \Rightarrow f * g \in L^1(\mathbb{R}^n)$

(ii) $f * g = g * f$

(iii) $(f * g) * h = f * (g * h)$

17. Exchanging limits and integration (25 points)

(i) Let $g \in \mathcal{C}^1\big(\mathbb{R}, L^1(\mathbb{R}^n)\big)$ a parameter-dependent function with values in $L^1(\mathbb{R}^n)$ so that there exists $h \in L^1(\mathbb{R}^n)$ with $\big|\partial_\lambda g(\lambda, x)\big| \leq h(x)$ for almost all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$. Show that differentiation with respect to λ and integration commute, i. e.

$$\frac{\partial}{\partial \lambda} \int_{\mathbb{R}^n} \mathrm{d}x \, g(\lambda, x) = \int_{\mathbb{R}^n} \mathrm{d}x \, \partial_\lambda g(\lambda, x) \, .$$

(ii) In addition to $f \in L^1(\mathbb{R}^n)$ assume $\|x_1 \, f\|_1 < \infty$. Then show that

$$\big(\mathrm{i}\partial_{\xi_1} \mathcal{F} f\big)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}x\cdot\xi} \, x_1 \, f(x)$$

holds.

(iii) In addition to $f, g \in L^1(\mathbb{R}^n)$ assume $\partial_{x_1} f, \partial_{x_1} g \in L^\infty(\mathbb{R}^n)$. Then show that

$$\partial_{x_1}(f * g) = \partial_{x_1} f * g = f * \partial_{x_1} g$$

holds.

18. Solving the heat equation using the convolution (22 points)

Consider the heat equation

$$\partial_t u(t,x) = D\partial_x^2 u(t,x), \qquad u(0,x) = f(x), \tag{1}$$

on $\mathbb{R}$ for $D > 0$. We will always assume $f \in L^1(\mathbb{R})$. For $t > 0$, define the function

$$G(t,x) := \frac{1}{\sqrt{4\pi D t}} \, \mathrm{e}^{-\frac{x^2}{4Dt}} \, .$$

(i) Show that $u(t)$ is integrable, i. e. $u(t) \in L^1(\mathbb{R})$ holds for all $t \in \mathbb{R}$.

(ii) Show that the function $u(t) := G(t) * f$ solves (1).
(You may use $\lim_{t \searrow 0} G(t) * f = f$ for all $f \in L^1(\mathbb{R}^n)$ without proof.)

(iii) Show that for any $t > 0$, the solution $u(t)$ is smooth in x.

(iv) Show that for any $x \in \mathbb{R}$, $\lim_{t \to \infty} u(t,x) = 0$.

Hand in home work on: Thursday, 17 October 2013, before class