UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Solutions 6
(2014.10.17)

Selfadjoint Operators

Homework Problems

20. Equivalent conditions for unitarity (19 points)

Prove the following statements:

(i) Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$. If $\langle A\varphi, \varphi \rangle = 0$ holds for all $\varphi \in \mathcal{H}$, then $A = 0$.

Hint: Consider the linear combination $\lambda\,\varphi + \mu\,\psi$ for various values of $\lambda, \mu \in \mathbb{C}$.

(ii) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then the following are equivalent:

(1) U is unitary, i. e. $U^* = U^{-1} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.

(2) $U\mathcal{H}_1 = \mathcal{H}_2$ and $\langle \varphi, \psi \rangle_{\mathcal{H}_1} = \langle U\varphi, U\psi \rangle_{\mathcal{H}_2}$ for all $\varphi, \psi \in \mathcal{H}_1$.

(3) $U\mathcal{H}_1 = \mathcal{H}_2$ and $\lVert U\varphi \rVert_{\mathcal{H}_2} = \lVert \varphi \rVert_{\mathcal{H}_1}$ for all $\varphi \in \mathcal{H}_1$.

(iii) Give an example of a map $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ which is not unitary even though $\langle \varphi, \psi \rangle_{\mathcal{H}_1} = \langle U\varphi, U\psi \rangle_{\mathcal{H}_2}$ is satisfied for all $\varphi, \psi \in \mathcal{H}_1$. Why does that example not contradict the equivalences from (ii)?

Solution:

(i) By the assumption, we have

$$\begin{aligned} 0 &\overset{[1]}{=} \Big\langle A\big(\lambda\,\varphi + \mu\,\psi\big), \big(\lambda\,\varphi + \mu\,\psi\big) \Big\rangle - \lvert\lambda\rvert^2 \,\langle A\varphi, \varphi \rangle - \lvert\mu\rvert^2 \,\langle A\psi, \psi \rangle \\ &\overset{[1]}{=} \overline{\lambda}\mu \,\langle A\varphi, \psi \rangle + \lambda\overline{\mu} \,\langle A\psi, \varphi \rangle \end{aligned}$$

for all values of $\lambda, \mu \in \mathbb{C}$. Setting $\lambda = 1 = \mu$ [1] yields

$$\langle A\varphi, \psi \rangle + \langle A\psi, \varphi \rangle \overset{[1]}{=} 0,$$

and choosing $\lambda = \mathrm{i}$ and $\mu = 1$ [1] yields

$$\mathrm{i}\,\langle A\varphi, \psi \rangle - \mathrm{i}\,\langle A\psi, \varphi \rangle \overset{[1]}{=} 0.$$

Therefore, we obtain two equations with two unknowns ($\langle A\varphi, \psi \rangle$ and $\langle A\psi, \varphi \rangle$), and solving this system of equations yields $\langle A\varphi, \psi \rangle = 0$ for all $\varphi, \psi \in \mathcal{H}$. This is the case if and only if $A = 0$ [1].

(ii) "(1) $\Rightarrow$ (2):" Assume U is unitary. Then $U^{-1} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ immediately implies $U\mathcal{H}_1 = \mathcal{H}_2$ [1], and it follows from $U^* = U^{-1}$ that

$$\left\langle U\varphi, U\psi \right\rangle_{\mathcal{H}_2} \overset{[1]}{=} \left\langle U^* U\varphi, \psi \right\rangle_{\mathcal{H}_1} \overset{[1]}{=} \langle \varphi, \psi \rangle_{\mathcal{H}_1} ,$$

and we have shown (2).

"(2) $\Rightarrow$ (3):" This is evident, just pick $\varphi = \psi$ [1].

"(3) $\Rightarrow$ (1):" Suppose U satisfies $U\mathcal{H}_1 = \mathcal{H}_2$ and $\|U\varphi\|_{\mathcal{H}_2} = \|\varphi\|_{\mathcal{H}_1}$ for all $\varphi \in \mathcal{H}_1$. Then also

$$\left\langle U^* U\varphi, \varphi \right\rangle_{\mathcal{H}_1} \overset{[1]}{=} \left\langle U\varphi, U\varphi \right\rangle_{\mathcal{H}_2} \overset{[1]}{=} \langle \varphi, \varphi \rangle_{\mathcal{H}_1}$$

holds true for all $\varphi \in \mathcal{H}_1$. Thus, we deduce $U^* U - \mathrm{id}_{\mathcal{H}_1} = 0$ with the help of (i) [1], and U is unitary by definition [1].

(iii) Pick $U = T_t$, $t > 0$, on $L^2\big([0, +\infty)\big)$ as defined in problem 16 [1]. Then even though we have $\langle T_t\varphi, T_t\psi \rangle = \langle \varphi, \psi \rangle$, the adjoint T_t^* is not the inverse T_t^{-1} [1]. However, this is not in contradiction to (ii), because $T_t L^2\big([0, +\infty)\big) \subsetneq L^2\big([0, +\infty)\big)$ [1].

21. Translations on the interval and its generator (17 points)

Consider the problem of translations on $L^2([0,1])$ and their generators from Chapter 4.3.2. We will reuse all of the notation, e. g. P_{min} is the operator $-\mathrm{i}\partial_x$ equipped with domain

$$\mathcal{D}_{\text{min}} := \bigl\{\varphi \in L^2([0,1]) \;\mid\; -\mathrm{i}\partial_x\varphi \in L^2([0,1]),\; \varphi(0) = 0 = \varphi(1)\bigr\}.$$

(i) Show that $P_{\text{min}}^* = P_{\text{max}}$.

(ii) Compute the deficiency indices for P_{min}.

(iii) Show that $P_\vartheta = P_\vartheta^*$ is selfadjoint.

Solution:

(i) Clearly, $P_{\text{min}}^* \subseteq P_{\text{max}}$, because $-\mathrm{i}\partial_x\varphi \in L^2([0,1])$ is a necessary condition [1]. Now let $\psi \in \mathcal{D}_{\text{min}}$ and $\varphi \in \mathcal{D}_{\text{max}}$. Then a quick computation reveals

$$\begin{aligned}\bigl\langle \varphi, P_{\text{min}}\psi\bigr\rangle &\overset{[1]}{=} \int_0^1 \mathrm{d}x\, \overline{\varphi(x)}\bigl(-\mathrm{i}\partial_x\psi\bigr)(x) \\ &\overset{[1]}{=} -\mathrm{i}\,\Bigl[\overline{\varphi(x)}\,\psi(x)\Bigr]_0^1 + \int_0^1 \mathrm{d}x\, \overline{\bigl(-\mathrm{i}\partial_x\varphi\bigr)(x)}\,\psi(x) \\ &\overset{[1]}{=} \bigl\langle -\mathrm{i}\partial_x\varphi, \psi\bigr\rangle \overset{[1]}{=} \bigl\langle P_{\text{max}}\varphi, \psi\bigr\rangle,\end{aligned}$$

and $\psi \in \mathcal{D}_{\text{min}}$ suffices to make the boundary terms vanish. Consequently, we have shown $P_{\text{min}}^* = P_{\text{max}}$ [1].

(ii) It is clear that $-\mathrm{i}\partial_x \mathrm{e}^{\pm x} = \mp\mathrm{i}\mathrm{e}^{\pm\varphi}$ are the only two solutions (up to scalar multiples, of course) [1], and that $\mathrm{e}^{\pm x} \in L^2([0,1])$ [1]. Moreover, $\varphi_\pm \in \mathcal{D}_{\text{max}}$ because their derivative are again L^2 [1]. That means $N_\pm(P_{\text{min}}) = 1 \neq 0$ and $\ker\bigl(P_{\text{min}} \pm \mathrm{i}\bigr) \neq \{0\}$ is non-trivial [1].

(iii) Clearly, as a symmetric operator, P_ϑ^* is densely defined, and $P_\vartheta^* \subseteq P_{\text{max}}$ for the same reasons as in (i) [1]. A computation analogous to that in (i) [1] reveals that $\mathcal{D}(P_\vartheta^*)$ must consist of vectors so that

$$\overline{\varphi(0)}\,\psi(0) = \overline{\varphi(1)}\,\psi(1) = \overline{\varphi(1)}\,\mathrm{e}^{+\mathrm{i}\vartheta}\psi(0) \qquad [1]$$

holds for all $\psi \in \mathcal{D}_\vartheta$ [1]. Solving for the boundary condition of φ, we obtain $\varphi(0) = \mathrm{e}^{-\mathrm{i}\vartheta}\varphi(1)$ [1], and hence, $\varphi \in \mathcal{D}_\vartheta$ [1]. That means $\mathcal{D}(P_\vartheta^*) = \mathcal{D}_\vartheta$, and we have shown the selfadjointness of $P_\vartheta^* = P_\vartheta$ [1].

22. Translations on the half line (18 points)

Consider the Hilbert space $L^2\big([0,+\infty)\big)$.

(i) Show that there exists no selfadjoint extension of $P = -\mathrm{i}\partial_x$ with domain $\mathcal{D}(P) = \mathcal{C}^\infty_{\mathrm{c}}\big((0,+\infty)\big)$.

(ii) Why does (i) imply that there cannot be a unitary evolution group associated to translations on $L^2\big([0,+\infty)\big)$?

Solution:

(i) We need to compute the deficiency indices: as before, the equation

$$-\mathrm{i}\partial_x \varphi_\pm = \mp \mathrm{i} \varphi_\pm \qquad [1]$$

has $\varphi_\pm(x) = \mathrm{e}^{\pm x}$ as its only non-trivial solution (up to a scalar multiple) [1]. However, only $\varphi_-(x) = \mathrm{e}^{-x} \in L^2\big([0,+\infty)\big)$ is square-integrable [1], so we already know $\ker\big(P^* + \mathrm{i}\big) = \{0\}$ and $N_+(P) = 0$ [1].

Now we need to check whether $\varphi_- \in \mathcal{D}(P^*)$ [1]: the domain of the adjoint is surely contained in

$$\mathcal{D}_{\mathrm{max}} \overset{[1]}{=} \Big\{ \varphi \in L^2\big([0,+\infty)\big) \;\big|\; -\mathrm{i}\partial_x \varphi \in L^2\big([0,+\infty)\big) \Big\}.$$

A short computation reveals that this is indeed enough, and $\mathcal{D}(P^*) = \mathcal{D}_{\mathrm{max}}$: let $\varphi \in \mathcal{D}(P^*) = \mathcal{D}_{\mathrm{max}}$ and $\varphi \in \mathcal{D}(P)$, then partial integration yields

$$\begin{aligned} \big\langle \varphi, P\psi \big\rangle &\overset{[1]}{=} \int_0^{+\infty} \mathrm{d}x \, \overline{\varphi(x)} \, \big(-\mathrm{i}\partial_x \psi\big)(x) \\ &\overset{[1]}{=} -\mathrm{i} \Big[\overline{\varphi(x)} \, \psi(x) \Big]_0^1 + \int_0^{+\infty} \mathrm{d}x \, \overline{\big(-\mathrm{i}\partial_x \varphi\big)(x)} \, \psi(x) \\ &\overset{[1]}{=} \big\langle -\mathrm{i}\partial_x \varphi, \psi \big\rangle \overset{[1]}{=} \big\langle P^* \varphi, \psi \big\rangle, \end{aligned}$$

because $\psi(0) = 0$ for all smooth functions whose compact support lies in the open set $(0,+\infty)$ [1]. That means $\varphi_- \in \mathcal{D}(P^*)$ [1] and $N_-(P) = \dim\ker\big(P^* - \mathrm{i}\big) = 1$ [1].

Since $N_-(P) = 1 \neq 0 = N_+(P)$ [1], we deduce form Theorem 5.2.7 that there exists no selfadjoint extension of P [1].

(ii) By Stone's Theorem, unitary evolution groups are in one-to-one correspondence with selfadjoint operators [1]. And since there exists no selfadjoint extension of P, one cannot define a unitary evolution group of translations either [1].

23. The radial part of the Laplace operator in $d = 3$ (19 points)

Consider the radial part of $-\Delta_x$ on $L^2(\mathbb{R}^3)$, the operator

$$H_{\text{rad}} = -\tfrac{1}{2}\partial_r^2 - \tfrac{1}{r}\partial_r,$$

with domain $\mathcal{C}_c^\infty\big((0,+\infty)\big)$ on the Hilbert space $L^2\big([0,+\infty)\big)$ endowed with the scalar product

$$\langle \varphi, \psi \rangle = \int_0^{+\infty} \mathrm{d}r\, r^2\, \overline{\varphi(r)}\, \psi(r).$$

(The factor r^2 stems from $\int_{\mathbb{R}^3} \mathrm{d}x = \int_0^{2\pi} \mathrm{d}\varphi \int_{-\pi}^{+\pi} \mathrm{d}\vartheta \int_0^{+\infty} \mathrm{d}r\, r^2\, \sin\vartheta$ in spherical coordinates.)

(i) Show that H_{rad} is symmetric.

(ii) Find out whether H_{rad} is essentially selfadjoint.

Solution:

(i) All of the boundary terms in the following computation vanish [1], because $\varphi, \psi \in \mathcal{C}_c^\infty\big((0,+\infty)\big)$ are smooth, and their supports are compact and do not contain 0. Moreover, all the integrals below exist. Then partially integrating twice yields

$$\begin{aligned}
\big\langle \varphi, H_{\text{rad}}\psi \big\rangle &\overset{[1]}{=} -\frac{1}{2}\int_0^{+\infty} \mathrm{d}r\, r^2\, \overline{\varphi(r)}\, \big(\partial_r^2\psi(r) + 2r^{-1}\,\partial_r\psi(r)\big) \\
&\overset{[1]}{=} -\frac{1}{2}\Big[r^2\,\overline{\varphi(r)}\,\partial_r\psi(r) + 2r\,\overline{\varphi(r)}\,\psi(r)\Big]_0^{+\infty} + \\
&\qquad + \frac{1}{2}\int_0^{+\infty} \mathrm{d}r\, \Big(\partial_r\big(r^2\,\overline{\varphi(r)}\big)\,\partial_r\psi(r) + \partial_r\big(2r\,\overline{\varphi(r)}\big)\,\psi(r)\Big) \\
&\overset{[1]}{=} \frac{1}{2}\int_0^{+\infty} \mathrm{d}r\, \Big(2r\,\overline{\varphi(r)}\,\partial_r\psi(r) + r^2\,\overline{\partial_r\varphi(r)}\,\partial_r\psi(r) + 2\,\overline{\varphi(r)}\,\psi(r) + 2r\,\overline{\partial_r\varphi(r)}\,\psi(r)\Big) \\
&\overset{[1]}{=} \frac{1}{2}\Big[2r\,\overline{\varphi(r)}\,\psi(r) + r^2\,\overline{\partial_r\varphi(r)}\,\psi(r)\Big]_0^{+\infty} + \\
&\qquad - \frac{1}{2}\int_0^{+\infty} \mathrm{d}r\, \Big(\partial_r\big(2r\,\overline{\varphi(r)}\big) + \partial_r\big(r^2\,\overline{\partial_r\varphi(r)}\big) - 2\,\overline{\varphi(r)} - 2r\,\overline{\partial_r\varphi(r)}\Big)\,\psi(r) \\
&\overset{[1]}{=} -\frac{1}{2}\int_0^{+\infty} \mathrm{d}r\, r^2\, \big(\overline{\partial_r^2\varphi(r)} + 2r^{-1}\,\overline{\partial_r\varphi(r)}\big)\,\psi(r) \\
&\overset{[1]}{=} \big\langle H_{\text{rad}}\varphi, \psi \big\rangle.
\end{aligned}$$

Hence, H_{rad} is symmetric [1].

(ii) By the Fundamental Criterion, we have have to check whether the deficiency indices $N_\pm(H_{\text{rad}}) = 0$ are zero [1], i. e. we need to solve

$$-\tfrac{1}{2}\partial_r^2\varphi_\pm - r^{-1}\,\partial_r\varphi_\pm = \mp\mathrm{i}\varphi_\pm. \qquad [1]$$

These equations have the non-zero solutions

$$\varphi_\pm(r) \overset{[2]}{=} \frac{\mathrm{e}^{-(1\mp\mathrm{i})r}}{r}$$

which decay exponentially towards $+\infty$ for both choices of sign [1]. Moreover, the $1/r$ singularity at $r = 0$ is neutralized by the factor r^2 in the measure [1], and we deduce $\varphi_\pm \in$

$L^2\big([0,+\infty)\big)$ [1]. It remains to show that $\varphi_\pm \in \mathcal{D}(H^*_{\text{rad}})$ [1]: a quick inspection of the calculation in (i) confirms that $\psi \in \mathcal{C}^\infty_{\text{c}}\big((0,+\infty)\big)$ is enough to make the boundary terms vanish [1], and hence, $\varphi_\pm \in \mathcal{D}(H^*_{\text{rad}})$ holds.

That means $N_\pm(H_{\text{rad}}) \geq 1$ [1], and H_{rad} is not essentially selfadjoint by the Fundamental criterion [1].