The Heat Equation & Hilbert Spaces

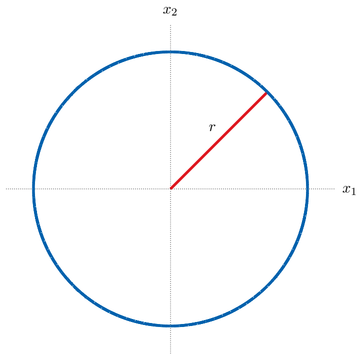
Homework Problems

19. The heat equation on a ring (17 points)

Assume a circular ring of radius r has been lying in a heat bath with temperature distribution $T(x_1, x_2) = T_0 \frac{x_1}{r}$, $T_0 > 0$, for a very long time.

At time $t = 0$, the ring is removed from the heat bath, and for $t > 0$ the temperature distribution $u(t, s)$, s being the arc length, satisfies the heat equation

$$\partial_t u = a^2 \, \partial_s^2 u \, , \qquad a > 0 \, .$$



(i) Compute $u(t, s)$ for $t > 0$ using separation of variables. (Hint: Use $u(t, s) \in \mathbb{R}$ to simplify your arguments.)

(ii) After what time has the maximal difference in temperature decreased to the $1/e$th fraction of that at time $t = 0$?

Solution:

(i) We first rewrite the initial condition in terms of the arc length $s = r\varphi$:

$$f(s) = T\big(r \cos \tfrac{s}{r}, r \sin \tfrac{s}{r}\big) = T_0 \, \frac{r \cos \frac{s}{r}}{r} \overset{[1]}{=} T_0 \, \cos \tfrac{s}{r}$$

If we identify the circle of radius r with the interval $[0, 2\pi r]$, then the solutions need to satisfy *periodic boundary conditions* [1],

$$u(t, 0) = u(t, 2\pi r) \, .$$

Equivalently, we can think of periodic functions on $\mathbb{R}$ where $u(t, s + 2\pi r) = u(t, s)$ holds. After plugging in the product ansatz $u(t, s) = \tau(t) \, \zeta(s)$ [1] into the heat equation,

$$\dot{\tau}(t) \, \zeta(x) \overset{[1]}{=} a^2 \, \tau(t) \, \zeta''(s) \, ,$$

we obtain two coupled ODEs,

$$\frac{1}{a^2} \frac{\dot{\tau}(t)}{\tau(t)} = \frac{\zeta''(s)}{\zeta(s)} = \lambda \in \mathbb{R} \, . \qquad [1]$$

Note that in this case, we may assume that λ is real instead of complex. Solving the equation for ζ yields

$$\zeta_\lambda(s) \overset{[1]}{=} \begin{cases} a_1(\lambda)\ \sin\sqrt{|\lambda|}s + a_2(\lambda)\ \cos\sqrt{|\lambda|}s & \lambda < 0 \\ a_1(0) + a_2(0)\, s & \lambda = 0 \\ a_1(\lambda)\ \sinh\sqrt{|\lambda|}s + a_2(\lambda)\ \cosh\sqrt{|\lambda|}s & \lambda > 0 \end{cases}$$

Imposing periodic boundary conditions eliminates the solutions for $\lambda < 0$ [1] and the linear solution for $\lambda = 0$ [1]. For $\lambda < 0$, only those solutions are admissible which have $2\pi r$-periodicity, i. e.

$$\lambda \overset{[1]}{=} -\left(\frac{n}{r}\right)^2, \qquad\qquad n \in \mathbb{N}_0 .$$

Now we can solve the second equation for those special λs,

$$\tau_n(t) \overset{[1]}{=} \tau(0)\,\mathrm{e}^{-a^2\frac{n^2}{r^2}t} .$$

Hence, any solution is a linear combination of the form

$$u(t,s) \overset{[1]}{=} \sum_{n=0}^{\infty} \mathrm{e}^{-a^2\frac{n^2}{r^2}t}\left(a_1(n)\ \sin\tfrac{ns}{r} + a_2(n)\ \cos\tfrac{ns}{r}\right)$$

To satisfy the initial condition, $u(0,s) = f(s)$, we set all but one equal to 0,

$$u(0,s) \overset{[1]}{=} \sum_{n=0}^{\infty}\left(a_1(n)\ \sin\tfrac{ns}{r} + a_2(n)\ \cos\tfrac{ns}{r}\right) \overset{!}{=} T_0\ \cos\tfrac{s}{r} ,$$

and thus the solution is

$$u(t,s) \overset{[1]}{=} T_0\,\mathrm{e}^{-\frac{a^2}{r^2}t}\ \cos\tfrac{s}{r} .$$

(ii) The maximal temperature difference is

$$\begin{aligned} \Delta u(t) &\overset{[1]}{:=} \max_{s\in[0,2\pi r]} u(t,s) - \min_{s\in[0,2\pi r]} u(t,s) \\ &= T_0\,\mathrm{e}^{-\frac{a^2}{r^2}t}\big(1-(-1)\big) \overset{[1]}{=} 2T_0\,\mathrm{e}^{-\frac{a^2}{r^2}t} . \end{aligned}$$

Now if we require that at t_*, the difference is $1/\mathrm{e}$th of the initial maximal temperature difference,

$$\frac{\Delta u(t_*)}{\Delta u(0)} \overset{!}{=} \frac{1}{\mathrm{e}} \overset{[1]}{=} \mathrm{e}^{-\frac{a^2}{r^2}t_*} ,$$

we obtain $t_* = \frac{r^2}{a^2}$ [1].

20. The Fourier basis on $L^2([-\pi,+\pi])$ (19 points)

Consider the Hilbert space of square integrable functions $L^2([-\pi,+\pi])$ endowed with the scalar product

$$\langle f, g\rangle_{L^2} := \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}x \, \overline{f(x)} \, g(x) \, .$$

(i) Show that $\{\mathrm{e}^{+inx}\}_{n\in\mathbb{Z}}$ is an orthonormal system.

(ii) Show that $\{1\} \cup \big\{\sqrt{2}\,\sin nx \, , \, \sqrt{2}\,\cos nx\big\}_{n\in\mathbb{N}}$ is an orthonormal system.

The orthonormal system $\{\mathrm{e}^{+inx}\}_{n\in\mathbb{Z}}$ is also an orthonormal *basis* of $L^2([-\pi,+\pi])$. Moreover, let $\ell^2(\mathbb{Z})$ be the Hilbert space of square summable sequences with scalar product

$$\langle a, b\rangle_{\ell^2} := \sum_{n\in\mathbb{Z}} \overline{a_n}\, b_n \, , \qquad a = (a_n)_{n\in\mathbb{Z}}, \; b = (b_n)_{n\in\mathbb{Z}} \, .$$

(iii) Show that for any $f \in L^2([-\pi,+\pi])$ we have $\sum_{n\in\mathbb{Z}} \big|\langle \mathrm{e}^{+inx}, f\rangle\big|_{L^2}^2 < \infty$.

(iv) Show that the map

$$\mathcal{F} : L^2([-\pi,+\pi]) \longrightarrow \ell^2(\mathbb{Z}), \; f \mapsto \mathcal{F}f := \Big(\big\langle \mathrm{e}^{+inx}, f\big\rangle_{L^2}\Big)_{n\in\mathbb{Z}}$$

is norm-preserving, i. e. $\|f\|_{L^2} = \|\mathcal{F}f\|_{\ell^2}$ holds for any $f \in L^2([-\pi,+\pi])$.

(v) Show that $\mathcal{F}$ is linear, i. e. for all $f, g \in L^2([-\pi,+\pi])$ and $\alpha \in \mathbb{C}$ we have

$$\mathcal{F}\big(\alpha\, f + g\big) = \alpha\, \mathcal{F}f + \mathcal{F}g \, .$$

(vi) Show that $\mathcal{F}$ is bijective.

Solution:

(i) The vectors e^{+inx} are normalized:

$$\big\langle \mathrm{e}^{+inx}, \mathrm{e}^{+inx}\big\rangle_{L^2} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}x \, \overline{\mathrm{e}^{+inx}} \, \mathrm{e}^{+inx} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}x = 1. \qquad [1]$$

For $n \neq k$, the vectors are orthogonal:

$$\begin{aligned}
\Big\langle \mathrm{e}^{+inx}, \mathrm{e}^{+ikx}\Big\rangle_{L^2} &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}x \, \overline{\mathrm{e}^{+inx}} \, \mathrm{e}^{+ikx} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}x \, \mathrm{e}^{+\mathrm{i}(k-n)x} \\
&\overset{[1]}{=} \left[\frac{1}{2\pi}\frac{1}{\mathrm{i}\,(k-n)}\,\mathrm{e}^{+\mathrm{i}(k-n)x}\right]_{-\pi}^{+\pi} = \frac{\mathrm{e}^{+\mathrm{i}(k-n)\pi} - \mathrm{e}^{-\mathrm{i}(k-n)\pi}}{\mathrm{i}\,2\pi(k-n)} \\
&= \frac{(-1)^{k-n} - (-1)^{k-n}}{\mathrm{i}\,2\pi(k-n)} \overset{[1]}{=} 0 \, .
\end{aligned}$$

Hence, $\{\mathrm{e}^{+inx}\}_{n\in\mathbb{Z}}$ is an orthonormal system.

(ii) We write $\sin nx$ and $\cos nx$ are real and imaginary part of e^{+inx} and use $1 = \mathrm{e}^0$ as well as (i):

$$\begin{aligned}
\Big\langle 1, \sqrt{2}\,\cos nx\Big\rangle_{L^2} &\overset{[1]}{=} \frac{1}{\sqrt{2}}\Big(\big\langle 1, \mathrm{e}^{+\mathrm{i}nx}\big\rangle_{L^2} + \big\langle 1, \mathrm{e}^{-\mathrm{i}nx}\big\rangle_{L^2}\Big) = 0 \\
\Big\langle 1, \sqrt{2}\,\sin nx\Big\rangle_{L^2} &= \frac{1}{\mathrm{i}\sqrt{2}}\Big(\big\langle 1, \mathrm{e}^{+\mathrm{i}nx}\big\rangle_{L^2} - \big\langle 1, \mathrm{e}^{-\mathrm{i}nx}\big\rangle_{L^2}\Big) = 0
\end{aligned}$$

Now let us verify that the cos nx, $n \geq 1$, functions are orthonormal:

$$\begin{aligned}
\left\langle \sqrt{2}\,\cos nx, \sqrt{2}\,\cos kx \right\rangle_{L^2} &= \frac{2}{4}\left\langle \mathrm{e}^{+\mathrm{i}nx} + \mathrm{e}^{-\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx} \right\rangle_{L^2} \\
&= \frac{1}{2}\Big(\big\langle \mathrm{e}^{+\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}kx} \big\rangle_{L^2} + \big\langle \mathrm{e}^{-\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}kx} \big\rangle_{L^2} + \\
&\qquad + \big\langle \mathrm{e}^{+\mathrm{i}nx}, \mathrm{e}^{-\mathrm{i}kx} \big\rangle_{L^2} + \big\langle \mathrm{e}^{-\mathrm{i}nx}, \mathrm{e}^{-\mathrm{i}kx} \big\rangle_{L^2} \Big) \\
&= \tfrac{1}{2}\big(\delta_{n,k} + \delta_{-n,-k}\big) \overset{[1]}{=} \delta_{n,k}
\end{aligned}$$

Similarly, we show that sin nx, $n \geq 1$, are orthonormal [1].

Moreover, cos nx and sin kx are always orthogonal:

$$\begin{aligned}
\left\langle \sqrt{2}\,\cos nx, \sqrt{2}\,\sin kx \right\rangle_{L^2} &= \frac{2}{\mathrm{i}4}\left\langle \mathrm{e}^{+\mathrm{i}nx} + \mathrm{e}^{-\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}kx} - \mathrm{e}^{-\mathrm{i}kx} \right\rangle_{L^2} \\
&= \frac{1}{2}\Big(\big\langle \mathrm{e}^{+\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}kx} \big\rangle_{L^2} + \big\langle \mathrm{e}^{-\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}kx} \big\rangle_{L^2} + \\
&\qquad - \big\langle \mathrm{e}^{+\mathrm{i}nx}, \mathrm{e}^{-\mathrm{i}kx} \big\rangle_{L^2} - \big\langle \mathrm{e}^{-\mathrm{i}nx}, \mathrm{e}^{-\mathrm{i}kx} \big\rangle_{L^2} \Big) \\
&= \tfrac{1}{2}\big(\delta_{n,k} - \delta_{-n,-k}\big) \overset{[1]}{=} 0
\end{aligned}$$

Hence, $\{1\} \cup \big\{\sin nx, \cos nx\big\}_{n\in\mathbb{N}}$ is an orthonormal set.

(iii) We now use that $\{\mathrm{e}^{+\mathrm{i}nx}\}_{n\in\mathbb{Z}}$ is an orthonormal *basis*, i. e. any $f \in L^2([-\pi, +\pi])$ can be written as Fourier series,

$$f = \sum_{n\in\mathbb{Z}} \big\langle \mathrm{e}^{+\mathrm{i}nx}, f \big\rangle_{L^2}\, \mathrm{e}^{+\mathrm{i}nx},$$

where the right-hand side converges in the L^2-sense, i. e. $f_N := \sum_{|n|\leq N} \big\langle \mathrm{e}^{+\mathrm{i}nx}, f \big\rangle_{L^2}\, \mathrm{e}^{+\mathrm{i}nx}$ converges to f as $N \to \infty$,

$$\lim_{N\to\infty} \lVert f - f_N \rVert_{L^2}^2 = 0\,.$$

Thus, $\lVert f \rVert_{L^2}^2 < \infty$ can be written as

$$\begin{aligned}
\lVert f \rVert_{L^2}^2 &= \langle f, f \rangle_{L^2} \\
&\overset{[1]}{=} \Big\langle \textstyle\sum_{n\in\mathbb{Z}} \big\langle \mathrm{e}^{+\mathrm{i}nx}, f \big\rangle_{L^2}\, \mathrm{e}^{+\mathrm{i}nx}, \sum_{k\in\mathbb{Z}} \big\langle \mathrm{e}^{+\mathrm{i}kx}, f \big\rangle_{L^2}\, \mathrm{e}^{+\mathrm{i}kx} \Big\rangle_{L^2} \\
&\overset{[1]}{=} \sum_{n,k\in\mathbb{Z}} \overline{\big\langle \mathrm{e}^{+\mathrm{i}nx}, f \big\rangle}\, \big\langle \mathrm{e}^{+\mathrm{i}kx}, f \big\rangle\, \underbrace{\big\langle \mathrm{e}^{+\mathrm{i}nx}, \mathrm{e}^{+\mathrm{i}kx} \big\rangle}_{=\delta_{n,k}} \\
&\overset{[1]}{=} \sum_{n\in\mathbb{Z}} \big| \big\langle \mathrm{e}^{+\mathrm{i}nx}, f \big\rangle \big|^2 < \infty\,.
\end{aligned}$$

The right-hand side is finite as $f \in L^2([-\pi, +\pi])$ and thus, by definition $\lVert f \rVert < \infty$.

(iv) For any $f \in L^2([-\pi, +\pi])$, we compute

$$\begin{aligned}
\lVert \mathcal{F} f \rVert_{\ell^2}^2 &\overset{[1]}{=} \Big\lVert \Big(\big\langle \mathrm{e}^{+\mathrm{i}nx}, f \big\rangle_{L^2} \Big)_{n\in\mathbb{Z}} \Big\rVert_{\ell^2} \\
&\overset{[1]}{=} \sum_{n\in\mathbb{Z}} \big| \big\langle \mathrm{e}^{+\mathrm{i}nx}, f \big\rangle \big|^2 \overset{(iv)}{=} \lVert f \rVert_{L^2}^2\,. && [1]
\end{aligned}$$

The right-hand side is finite, because it coincides with $\lVert f \rVert_{L^2} < \infty$.

(v) Pick any $f, g \in L^2([-\pi, +\pi])$ and $\alpha \in \mathbb{C}$. Then we have:

$$\begin{aligned}\mathcal{F}(\alpha\, f + g) &\overset{[1]}{=} \Big(\big\langle \mathrm{e}^{+inx}, \alpha\, f + g\big\rangle_{L^2}\Big)_{n\in\mathbb{Z}} \overset{*}{=} \Big(\alpha\, \big\langle \mathrm{e}^{+inx}, f\big\rangle_{L^2} + \big\langle \mathrm{e}^{+inx}, g\big\rangle_{L^2}\Big)_{n\in\mathbb{Z}} \\ &= \alpha\, \Big(\big\langle \mathrm{e}^{+inx}, f\big\rangle_{L^2}\Big)_{n\in\mathbb{Z}} + \Big(\big\langle \mathrm{e}^{+inx}, g\big\rangle_{L^2}\Big)_{n\in\mathbb{Z}} \\ &\overset{[1]}{=} \alpha\, \mathcal{F}f + \mathcal{F}g\,.\end{aligned}$$

In the step marked with $*$, we have used the linearity of the scalar product in the second argument.

(vi) *Injectivity:* $\mathcal{F}f = 0$ implies $f = 0$, because $\mathcal{F}$ is norm-preserving. Hence, $\mathcal{F}$ is injective [1].

Surjectivity: Pick any $c \in \ell^2(\mathbb{Z})$; thus $\|c\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$. Define the function

$$f_c := \sum_{n\in\mathbb{Z}} c_n\, \mathrm{e}^{+inx}\,.$$

By definition, $\mathcal{F}f_c = c \in \ell^2(\mathbb{Z})$ holds [1], and thus the norm-preserving property (iv) yields that $f_c \in L^2([-\pi, +\pi])$,

$$\|f_c\|_{L^2}^2 = \big\|\mathcal{F}f_c\big\|_{\ell^2}^2 = \|c\|_{\ell^2}^2 < \infty\,. \qquad [1]$$

This means $\mathcal{F}$ is also surjective, and hence, bijective [1].

21. Orthogonal subspaces and projections onto subspaces (16 points)

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis (ONB) of a Hilbert space $\mathcal{H}$ and $N \in \mathbb{N}$.

(i) Prove that $E := \big\{\varphi_1, \ldots, \varphi_N\big\}^{\perp}$ is a sub *vector* space.

(ii) Give an ONB for the subspace $E = \big\{\varphi_1, \ldots, \varphi_N\big\}^{\perp}$.

(iii) Show that $\big(\{\varphi_1, \ldots, \varphi_N\}^{\perp}\big)^{\perp} = E^{\perp} = \mathrm{span}\{\varphi_1, \ldots, \varphi_N\}$.

Moreover, define the map

$$P : \mathcal{H} \longrightarrow \mathcal{H}\,,\ P\psi := \sum_{n=1}^{N} \langle \varphi_n, \psi\rangle\ \varphi_n\,.$$

(iv) Show that P is linear, i. e. for any $\varphi, \psi \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, we have $P\big(\alpha\varphi + \psi\big) = \alpha\, P\varphi + P\psi$.

(v) Show that P is a projection, i. e. $P^2 = P$.

(vi) Show that P is bounded, i. e. $\|P\varphi\| \leq \|\varphi\|$ holds for any $\varphi \in \mathcal{H}$.

Solution:

(i) The orthogonal complement is defined as

$$E \overset{[1]}{=} \big\{\psi \in \mathcal{H} \mid \langle \varphi_j, \psi\rangle = 0,\ j = 1, \ldots, N\big\}\,.$$

For any $\phi, \psi \in E$ and $\alpha \in \mathbb{C}$, also the vector $\alpha\,\phi + \psi$ is an element of E [1]: for all $j = 1, \ldots, N$

$$\langle \varphi_j, \alpha\,\phi + \psi\rangle = \alpha\ \langle \varphi_j, \phi\rangle + \langle \varphi_j, \psi\rangle \overset{[1]}{=} 0$$

is satisfied. Hence, E is a linear subspace of $\mathcal{H}$.

(ii) $\{\varphi_j\}_{j=N+1}^{\infty}$ [1]

(iii) Then $\varphi_j \in E^{\perp}$, because by definition of E

$$\langle \varphi_j, \psi\rangle \overset{[1]}{=} 0$$

holds for all $\psi \in \mathcal{H}$. Thus, $\varphi_j \in E^{\perp}$ for all $j = 1, \ldots, N$. By (i), E is a linear sub space [1].

Now assume that there exists a $\psi \in E^{\perp}$ which is *not* a linear combination of $\{\varphi_1, \ldots, \varphi_N\}$ [1]. Since $\big\{\varphi_j\big\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, we can express ψ as

$$\psi = \sum_{j=1}^{\infty} c_j\, \varphi_j\,. \qquad [1]$$

By assumption, there exists a $n \geq N + 1$ for which $c_n \neq 0$ [1]. But then

$$\langle \varphi_n, \psi\rangle = c_n \neq 0$$

and ψ cannot be an element of $E^{\perp}$, contradiction [1].

Hence, $E^{\perp} = \mathrm{span}\{\varphi_1, \ldots, \varphi_N\}$.

(iv) The linearity of P follows from the linearity of the scalar product in the first argument: for all $\phi, \psi \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} P\big(\alpha\,\phi + \psi\big) &\overset{[1]}{=} \sum_{j=1}^{N} \langle \varphi_j, \alpha\,\phi + \psi \rangle \; \varphi_j = \alpha \sum_{j=1}^{N} \langle \varphi_j, \phi \rangle \; \varphi_j + \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \; \varphi_j \\ &\overset{[1]}{=} \alpha\, P\phi + P\psi\,. \end{aligned}$$

Hence, P is linear.

(v) For any $\psi \in \mathcal{H}$, we deduce using the linearity of P:

$$\begin{aligned} P^2\psi = P\left(\sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \; \varphi_j \right) &\overset{[1]}{=} \sum_{k=1}^{N} \langle \varphi_j, \psi \rangle \; P\varphi_j \\ &= \sum_{k,j=1}^{N} \langle \varphi_j, \psi \rangle \; \underbrace{\langle \varphi_k, \varphi_j \rangle}_{=\delta_{k,j}} \, \varphi_k = \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \; \varphi_j \overset{[1]}{=} P\psi \end{aligned}$$

Hence, P is a projection.

(vi) With the help of Bessel's inequality [1], we obtain the claim:

$$\big\| P\psi \big\| = \left\| \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \; \varphi_j \right\| \overset{[1]}{\leq} \|\psi\|$$