Selfadjoint Operators

Homework Problems

20. Equivalent conditions for unitarity

Prove the following statements:

(i) Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$. If $\langle A\varphi, \varphi \rangle = 0$ holds for all $\varphi \in \mathcal{H}$, then $A = 0$.

Hint: Consider the linear combination $\lambda \, \varphi + \mu \, \psi$ for various values of $\lambda, \mu \in \mathbb{C}$.

(ii) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then the following are equivalent:

(1) U is unitary, i. e. $U^* = U^{-1} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.

(2) $U\mathcal{H}_1 = \mathcal{H}_2$ and $\langle \varphi, \psi \rangle_{\mathcal{H}_1} = \langle U\varphi, U\psi \rangle_{\mathcal{H}_2}$ for all $\varphi, \psi \in \mathcal{H}_1$.

(3) $U\mathcal{H}_1 = \mathcal{H}_2$ and $\lVert U\varphi \rVert_{\mathcal{H}_2} = \lVert \varphi \rVert_{\mathcal{H}_1}$ for all $\varphi \in \mathcal{H}_1$.

(iii) Give an example of a map $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ which is not unitary even though $\langle \varphi, \psi \rangle_{\mathcal{H}_1} = \langle U\varphi, U\psi \rangle_{\mathcal{H}_2}$ is satisfied for all $\varphi, \psi \in \mathcal{H}_1$. Why does that example not contradict the equivalences from (ii)?

21. Translations on the interval and its generator

Consider the problem of translations on $L^2([0,1])$ and their generators from Chapter 4.3.2. We will reuse all of the notation, e. g. P_{min} is the operator $-\mathrm{i}\partial_x$ equipped with domain

$$\mathcal{D}_{\mathrm{min}} := \bigl\{ \varphi \in L^2([0,1]) \;\big|\; -\mathrm{i}\partial_x \varphi \in L^2([0,1]), \; \varphi(0) = 0 = \varphi(1) \bigr\}.$$

(i) Show that $P_{\mathrm{min}}^* = P_{\mathrm{max}}$.

(ii) Compute the deficiency indices for P_{min}.

(iii) Show that $P_\vartheta = P_\vartheta^*$ is selfadjoint.

22. Translations on the half line

Consider the Hilbert space $L^2\bigl([0,+\infty)\bigr)$.

(i) Show that there exists no selfadjoint extension of $P = -\mathrm{i}\partial_x$ with domain $\mathcal{D}(P) = \mathcal{C}^\infty_{\mathrm{c}}\bigl((0,+\infty)\bigr)$.

(ii) Why does (i) imply that there cannot be a unitary evolution group associated to translations on $L^2\bigl([0,+\infty)\bigr)$?

23. The radial part of the Laplace operator in $d = 3$

Consider the radial part of $-\Delta_x$ on $L^2(\mathbb{R}^3)$, the operator

$$H_{\mathrm{rad}} = -\tfrac{1}{2}\partial_r^2 - \tfrac{1}{r}\partial_r,$$

with domain $\mathcal{C}_{\mathrm{c}}^{\infty}\big((0,+\infty)\big)$ on the Hilbert space $L^2\big([0,+\infty)\big)$ endowed with the scalar product

$$\langle \varphi, \psi \rangle = \int_0^{+\infty} \mathrm{d}r \, r^2 \, \overline{\varphi(r)} \, \psi(r).$$

(The factor r^2 stems from $\int_{\mathbb{R}^3} \mathrm{d}x = \int_0^{2\pi} \mathrm{d}\varphi \int_{-\pi}^{+\pi} \mathrm{d}\vartheta \int_0^{+\infty} \mathrm{d}r \, r^2 \, \sin\vartheta$ in spherical coordinates.)

(i) Show that H_{rad} is symmetric.

(ii) Find out whether H_{rad} is essentially selfadjoint.

Hand in home work on: Friday, 24 October 2014, before class