UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 6
(2013.10.17)

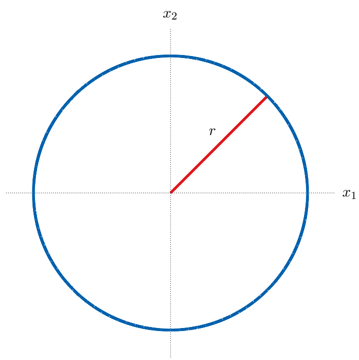
The Heat Equation & Hilbert Spaces

Homework Problems

19. The heat equation on a ring (17 points)

Assume a circular ring of radius r has been lying in a heat bath with temperature distribution $T(x_1, x_2) = T_0 \frac{x_1}{r}$, $T_0 > 0$, for a very long time.
At time $t = 0$, the ring is removed from the heat bath, and for $t > 0$ the temperature distribution $u(t, s)$, s being the arc length, satisfies the heat equation

$$\partial_t u = a^2 \, \partial_s^2 u \, , \qquad a > 0 \, .$$



(i) Compute $u(t, s)$ for $t > 0$ using separation of variables. (Hint: Use $u(t, s) \in \mathbb{R}$ to simplify your arguments.)

(ii) After what time has the maximal difference in temperature decreased to the $1/e$th fraction of that at time $t = 0$?

20. The Fourier basis on $L^2([-\pi, +\pi])$ (19 points)

Consider the Hilbert space of square integrable functions $L^2([-\pi, +\pi])$ endowed with the scalar product

$$\langle f, g \rangle_{L^2} := \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathrm{d}x \, \overline{f(x)} \, g(x) \, .$$

(i) Show that $\{\mathrm{e}^{+inx}\}_{n \in \mathbb{Z}}$ is an orthonormal system.

(ii) Show that $\{1\} \cup \{\sqrt{2} \, \sin nx \, , \, \sqrt{2} \, \cos nx\}_{n \in \mathbb{N}}$ is an orthonormal system.

The orthonormal system $\{\mathrm{e}^{+inx}\}_{n \in \mathbb{Z}}$ is also an orthonormal *basis* of $L^2([-\pi, +\pi])$. Moreover, let $\ell^2(\mathbb{Z})$ be the Hilbert space of square summable sequences with scalar product

$$\langle a, b \rangle_{\ell^2} := \sum_{n \in \mathbb{Z}} \overline{a_n} \, b_n \, , \qquad a = (a_n)_{n \in \mathbb{Z}}, \; b = (b_n)_{n \in \mathbb{Z}} \, .$$

(iii) Show that for any $f \in L^2([-\pi, +\pi])$ we have $\sum_{n \in \mathbb{Z}} \bigl| \langle \mathrm{e}^{+inx}, f \rangle \bigr|_{L^2}^2 < \infty$.

(iv) Show that the map

$$\mathcal{F} : L^2([-\pi, +\pi]) \longrightarrow \ell^2(\mathbb{Z}), \; f \mapsto \mathcal{F} f := \Bigl(\langle \mathrm{e}^{+inx}, f \rangle_{L^2} \Bigr)_{n \in \mathbb{Z}}$$

is norm-preserving, i. e. $\| f \|_{L^2} = \| \mathcal{F} f \|_{\ell^2}$ holds for any $f \in L^2([-\pi, +\pi])$.

(v) Show that $\mathcal{F}$ is linear, i. e. for all $f, g \in L^2([-\pi, +\pi])$ and $\alpha \in \mathbb{C}$ we have

$$\mathcal{F}\big(\alpha\, f + g\big) = \alpha\, \mathcal{F} f + \mathcal{F} g \,.$$

(vi) Show that $\mathcal{F}$ is bijective.

21. Orthogonal subspaces and projections onto subspaces (16 points)

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis (ONB) of a Hilbert space $\mathcal{H}$ and $N \in \mathbb{N}$.

(i) Prove that $E := \big\{\varphi_1, \ldots, \varphi_N\big\}^{\perp}$ is a sub *vector* space.

(ii) Give an ONB for the subspace $E = \big\{\varphi_1, \ldots, \varphi_N\big\}^{\perp}$.

(iii) Show that $\big(\{\varphi_1, \ldots, \varphi_N\}^{\perp}\big)^{\perp} = E^{\perp} = \mathrm{span}\big\{\varphi_1, \ldots, \varphi_N\big\}$.

Moreover, define the map

$$P : \mathcal{H} \longrightarrow \mathcal{H}\,, \; P\psi := \sum_{n=1}^{N} \langle \varphi_n, \psi \rangle \; \varphi_n \,.$$

(iv) Show that P is linear, i. e. for any $\varphi, \psi \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, we have $P\big(\alpha\varphi + \psi\big) = \alpha\, P\varphi + P\psi$.

(v) Show that P is a projection, i. e. $P^2 = P$.

(vi) Show that P is bounded, i. e. $\|P\varphi\| \leq \|\varphi\|$ holds for any $\varphi \in \mathcal{H}$.

Hand in home work on: Thursday, 24 October 2013, before class