UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Solutions 7
(2014.01.26)

Review for Midterm

There is *no homework due next Friday*. The midterm covers everything from **Chapter 1 up to and including the bit on the Birman-Schwinger principle in Chapter 5.2.2** (i. e. pp. 1–72) as well as **exercise sheets 1–6**. The coordinates of the Midterm in space-time are

Friday, 31 October, 15–17, in room MP 134 (at the usual place and the usual time during class).

No aids of any kind are allowed. And please bring a *photo* and *student id*.
The most important topics are listed below, but please note that not all problems in the Midterm need to be included in this list.

Material from APM 351 and other prerequisites

(1) Please also refer to the course material of APM 351 (published here on my homepage) for preparation, in particular Chapters 1, 9 and 10 of the lecture notes are relevant. Have a look at previous exams.

(2) You are expected to know discrete and continuous Fourier transforms (see e. g. problem 14 and equation (4.3.1)), and that these maps define unitary operators between Hilbert spaces.

(3) You should know standard computational techniques from calculus (e. g. partial integration). You are not expected to know integration tables by heart.

The framework of quantum mechanics

(1) Revisit Chapter 2. Study up on the fundamentals of quantum mechanics (states, obersvables and dynamical equations) in both, Heisenberg and Schrödinger picture.

Exercise:

(i) What physical consequences does the selfadjointness of the Hamilton operator have?

(ii) What is the physical significance of the unitarity of $\mathrm{e}^{-\mathrm{i}tH}$?

(iii) Give states, observables and dynamical equation in the Heisenberg picture for the spin-$1/2$ system from Chapter 2.1.

(2) Revisit Heisenberg's Uncertainty Relation (Chapter 2.2).

Exercise: Review the scaling argument on p. 63 (especially equation (5.2.3)) and explain its connection to Heisenberg's uncertainty relation.

Hilbert spaces

(1) Review Chapter 3. How are the most common Hilbert spaces defined ($\mathbb{C}^K$, $L^2(\mathbb{R}^d)$, $\ell^2(\mathbb{Z}^d)$)?

Exercise: State how the Hilbert space $L^2(\mathbb{R}^d, \mathbb{C}^2)$ is defined. Explain the equivalences $L^2(\mathbb{R}^3, \mathbb{C}^2) \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \cong L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.

(2) Let $U_j \in \mathcal{B}(\mathcal{H}_j)$, $j = 1, 2$, be two bounded operators on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Describe how $U_1 \oplus U_2$ acts on $\psi = (\psi_1, \psi_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$. Show that if the U_j are unitary on $\mathcal{H}_j$, then also $U_1 \oplus U_2$ is a unitary on $\mathcal{H}_1 \oplus \mathcal{H}_2$.

(3) Let $U_j \in \mathcal{B}(\mathcal{H}_j)$, $j = 1, 2$, be two bounded operators on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Find out how $U_1 \otimes \mathrm{id}_{\mathcal{H}_2}$ and $\mathrm{id}_{\mathcal{H}_1} \otimes U_2$ act on $\psi_1 \otimes \psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Show that if U_1 is a unitary on $\mathcal{H}_1$, then $U_1 \otimes \mathrm{id}_{\mathcal{H}_2}$ is a unitary on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Bounded operators

(1) Review the zoology of bounded operators (cf. e. g. Definition 4.2.3), in particular the definition of selfadjoint, unitary and positive operators as well as orthogonal projections.

Exercise:

(i) Let $P = P^* = P^2 \neq 0$ be an orthogonal projection on a Hilbert space $\mathcal{H}$. Prove $P \geq 0$.

(ii) Look up the different characterizations of unitarity (problem 20).

(2) Find out how the discrete Laplacian Δ (problems 13–14) is defined and how this operator is diagonalized. Do eigenfunctions of Δ exist?

Exercise:

(i) Solve APM 351 sheet 10, problem 37. How does the solution change if you replace the polynomial of shifts $\mathrm{id}_{\ell^2(\mathbb{Z}^2)} + q_1 \mathfrak{s}_1 + q_2 \mathfrak{s}_2$ with another one?

(ii) Prove that the discrete Laplacian Δ is selfadjoint and bounded.

(3) Study the properties of *multiplication operators* and enumerate the most common examples of operators in physics (e. g. problems 11–12, and Chapter 5.2.2.2).

(4) Review the calculus of 2×2 matrices (problems 1–2).

Unbounded operators

(1) Review how *unbounded*, densely defined operators are properly defined.

(2) What is the definition of the adjoint of a densely defined, unbounded operator?

Example: Prove that $T \subseteq S$ implies $S^* \subseteq T^*$.

(3) Work out the distinction between *symmetric* and *selfadjoint* operators. Do all symmetric operators satisfy the characterizations of the Fundamental Criterion of (Essential) Selfadjointness? Study examples (e. g. translations on $L^2([0, 1])$ and $L^2\big([0, +\infty)\big)$).

(4) Investigate the different ways to decompose the spectrum of an operator (point spectrum, continuous spectrum, essential spectrum and discrete spectrum).

(5) Review Chapter 5.2.2.3. You should know how to apply the Birman-Schwinger principle, for instance. What are bound states and where do the corresponding energy eigenvalues typically appear in the spectrum? I do *not* expect you to be able to make a proof akin to Theorem 5.2.13.

(6) Study the connection between unitary evolution groups and selfadjoint operators (Chapter 4.3.2).

Exercises:

(i) Repeat various proofs of symmetry and selfajointness, e. g. for $-\mathrm{i}\partial_x$ on various domains or $H = -\Delta_x$ (see e. g. problems 19 and 21–23).

(ii) Consider the hamiltonian $H_\lambda = -\partial_x^2 - \lambda\, V$ on $L^2(\mathbb{R})$ describing a one-dimensional quantum particle subjected to an attractive quantum well $V(x) = 1_{|x| \leq 1}(x)$. Show that H_λ has a bound state for all $0 < \lambda < \lambda_*$ where $\lambda_* \ll 1$ is small enough. Compute the approximate value of the eigenvalue.

PS There will be no solutions.