UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics

(APM 351 Y)

2013–2014
Solutions 7
(2013.10.24)

Weighted Hilbert spaces, the free Maxwell equations & Operators

Homework Problems

22. Weighted L^2-spaces (16 points)

Let $\varepsilon \in L^\infty(\mathbb{R}^n)$ be a function bounded away from 0 and $+\infty$, i. e. there exist $c, C > 0$ such that

$$0 < c \leq \varepsilon(x) \leq C < +\infty$$

holds for almost all $x \in \mathbb{R}^n$. Define the weighted L^2-space $L^2_\varepsilon(\mathbb{R}^n)$ as the pre-Hilbert space with scalar product

$$\langle f, g \rangle_\varepsilon := \int_{\mathbb{R}^n} \mathrm{d}x \, \varepsilon(x) \, \overline{f(x)} \, g(x) \tag{1}$$

so that $\|f\|_\varepsilon := \sqrt{\langle f, f \rangle_\varepsilon} < \infty$.

(i) Show that $f \in L^2(\mathbb{R}^n)$ if and only if $f \in L^2_\varepsilon(\mathbb{R}^n)$.

(ii) Show that the map

$$U_\varepsilon : L^2(\mathbb{R}^n) \longrightarrow L^2_\varepsilon(\mathbb{R}^n) \, , \; f \mapsto \sqrt{\varepsilon} f \, ,$$

is norm-preserving, i. e. $\|f\|_\varepsilon = \|U_\varepsilon f\|_{L^2(\mathbb{R}^n)}$ holds for all $f \in L^2(\mathbb{R}^n)$.

(iii) Show that $L^2_\varepsilon(\mathbb{R}^n)$ is indeed a Hilbert space, i. e. prove that it is complete.

Solution:

(i) "⇒:" Let $f \in L^2(\mathbb{R}^n)$. Then by definition $\|f\| < \infty$, and hence also

$$\|f\|_\varepsilon^2 \overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \varepsilon(x) \, \bigl|f(x)\bigr|^2 \overset{[1]}{\leq} \int_{\mathbb{R}^3} \mathrm{d}x \, C \, \bigl|f(x)\bigr|^2 \overset{[1]}{=} C \|f\|^2 < \infty \, .$$

"⇐:" Now assume $f \in L^2_\varepsilon(\mathbb{R}^n)$. Since $0 < 1/\varepsilon(x) \leq 1/c < +\infty$, we deduce

$$\begin{aligned} \|f\|^2 &\overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \bigl|f(x)\bigr|^2 = \int_{\mathbb{R}^3} \mathrm{d}x \, \frac{\varepsilon(x)}{\varepsilon(x)} \, \bigl|f(x)\bigr|^2 \\ &\overset{[1]}{\leq} c^{-1} \int_{\mathbb{R}^3} \mathrm{d}x \, \varepsilon(x) \, \bigl|f(x)\bigr|^2 \overset{[1]}{=} c^{-1} \, \|f\|_\varepsilon^2 \, . \end{aligned}$$

(ii) Let $f \in L^2(\mathbb{R}^n)$. Then we compute

$$\begin{aligned} \big\| U_\varepsilon f \big\|^2_{L^2(\mathbb{R}^n)} &\overset{[1]}{=} \big\langle \sqrt{\varepsilon} f , \sqrt{\varepsilon} f \big\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \mathrm{d}x \, \big| \sqrt{\varepsilon}(x) \, f(x) \big|^2 \\ &\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}x \, \varepsilon(x) \, \big| f(x) \big|^2 = \langle f , f \rangle_\varepsilon \overset{[1]}{=} \| f \|^2_\varepsilon \, . \end{aligned}$$

Hence, U_ε is norm-preserving.

(iii) Let $\{f_j\}_{j \in \mathbb{N}}$ be a Cauchy sequence in $L^2_\varepsilon(\mathbb{R}^n)$ [1]. Since U_ε is norm-preserving and linear, it is also invertible (the arguments are the same as in problem 20 (vi)) [1]. Moreover, the inverse $U_\varepsilon^{-1} = U_{\varepsilon^{-1}} : L^2_\varepsilon(\mathbb{R}^n) \longrightarrow {}^2(\mathbb{R}^n)$ is also norm-preserving by (ii) [1].

That means $\big\{ U_{\varepsilon^{-1}} f_j \big\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ [1]. Seeing as $L^2(\mathbb{R}^n)$ is complete, $U_{\varepsilon^{-1}} f_j$ converges to some $g \in L^2(\mathbb{R}^n)$ [1]. But then f_j converges to $U_\varepsilon g$ in $L^2_\varepsilon(\mathbb{R}^n)$ [1],

$$\begin{aligned} \big\| f_j - U_\varepsilon g \big\|_\varepsilon &= \big\| U_\varepsilon \big(U_{\varepsilon^{-1}} f_j - g \big) \big\|_\varepsilon \\ &\overset{(ii)}{=} \big\| U_{\varepsilon^{-1}} f_j - g \big\|_{L^2(\mathbb{R}^n)} \xrightarrow{j \to \infty} 0 \, . \end{aligned}$$

Hence, $L^2_\varepsilon(\mathbb{R}^n)$ is complete, and thus a Hilbert space [1].

23. The free Maxwell equations as Schrödinger-type equation (17 points)

Consider the dynamical Maxwell equations

$$\begin{aligned} \partial_t \mathbf{E}(t) &= +\nabla_x \times \mathbf{H}(t)\,, & \mathbf{E}(0) &= \mathbf{E}^{(0)} \in L^2(\mathbb{R}^3,\mathbb{C}^3)\,, \\ \partial_t \mathbf{H}(t) &= -\nabla_x \times \mathbf{E}(t)\,, & \mathbf{H}(0) &= \mathbf{H}^{(0)} \in L^2(\mathbb{R}^3,\mathbb{C}^3)\,. \end{aligned} \tag{2}$$

Here, $L^2(\mathbb{R}^n,\mathbb{C}^N)$ is the Hilbert space with scalar product

$$\langle \Psi, \Phi \rangle := \int_{\mathbb{R}^3} \mathrm{d}x \, \Psi(x) \cdot \Phi(x)$$

defined in terms of the scalar product $\Psi(x) \cdot \Phi(x) := \sum_{j=1}^N \overline{\Psi_j(x)}\, \Phi_j(x)$ on $\mathbb{C}^N$.

Moreover, consider also the Schrödinger-type equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \mathbf{Rot} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \end{pmatrix} = \begin{pmatrix} \mathbf{E}^{(0)} \\ \mathbf{H}^{(0)} \end{pmatrix} \in L^2(\mathbb{R}^3,\mathbb{C}^6)\,, \tag{3}$$

where the free Maxwell operator

$$\mathbf{Rot} := \begin{pmatrix} 0 & +\mathrm{i}\nabla_x^{\times} \\ -\mathrm{i}\nabla_x^{\times} & 0 \end{pmatrix}$$

is defined in terms of the curl $\nabla_x^{\times}\mathbf{E} := \nabla_x \times \mathbf{E}$.

During the computations, you may work with Rot and $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ as if they were $n \times n$ matrices.

(i) Verify that $\big(\mathbf{E}(t),\mathbf{H}(t)\big) := \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} \big(\mathbf{E}^{(0)},\mathbf{H}^{(0)}\big)$ solves (3).

(ii) Show that the dynamical Maxwell equations (2) can be recast in the form (3).

(iii) Define complex conjugation C as $(C\Psi)(x) := \overline{\Psi(x)}$. Confirm that $C\,\mathbf{Rot}\,C = -\mathbf{Rot}$ holds.

(iv) Prove $C\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,C = \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ as well as that $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ commutes with complex conjugation, i. e.

$$\big[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, C\big] := \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,C - C\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} = 0\,.$$

(v) Show that $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ commutes with the real part operator $\mathrm{Re} := \frac{1}{2}(1+C)$, i. e. $\big[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, \mathrm{Re}\big] = 0$.

(vi) Show that if $\big(\mathbf{E}^{(0)},\mathbf{H}^{(0)}\big)$ is initially real-valued, then the solution $\big(\mathbf{E}(t),\mathbf{H}(t)\big)$ to the Maxwell equations is also real-valued.

Solution:

(i) By direct computation, we obtain that the ansatz solves the dynamical equation,

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} &= \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} \begin{pmatrix} \mathbf{E}^{(0)} \\ \mathbf{H}^{(0)} \end{pmatrix} \right) = -\mathrm{i}^2 \mathbf{Rot}\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} \begin{pmatrix} \mathbf{E}^{(0)} \\ \mathbf{H}^{(0)} \end{pmatrix} \\ &\overset{[1]}{=} \mathbf{Rot} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix}, \end{aligned}$$

and also satisfies the initial condition,

$$\big(\mathbf{E}(0),\mathbf{H}(0)\big) = \mathrm{e}^{0}\,\big(\mathbf{E}^{(0)},\mathbf{H}^{(0)}\big) \overset{[1]}{=} \big(\mathbf{E}^{(0)},\mathbf{H}^{(0)}\big)\,.$$

(ii) If we multiply equations (2) with i and arrange them as a vector in $\mathbb{C}^6$, we obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} \overset{[1]}{=} \begin{pmatrix}+\mathrm{i}\nabla_x \times \mathbf{H}(t)\\ -\mathrm{i}\nabla_x \times \mathbf{E}(t)\end{pmatrix}$$

On the other hand, computing

$$\mathbf{Rot}\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix} \overset{[1]}{=} \begin{pmatrix}+\nabla_x \times \mathbf{H}(t)\\ -\nabla_x \times \mathbf{E}(t)\end{pmatrix}$$

yields that the right-hand sides agree.

(iii) Plugging in the definition of complex conjugation and applying the operator to an arbitrary $(\mathbf{E}, \mathbf{H})$, we obtain

$$\begin{aligned} C\,\mathbf{Rot}\,C\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} &\overset{[1]}{=} C\,\mathbf{Rot}\begin{pmatrix}\overline{\mathbf{E}}\\ \overline{\mathbf{H}}\end{pmatrix} = C\begin{pmatrix}+\mathrm{i}\nabla_x \times \overline{\mathbf{H}}\\ -\mathrm{i}\nabla_x \times \overline{\mathbf{E}}\end{pmatrix} \\ &\overset{[1]}{=} \begin{pmatrix}\overline{+\mathrm{i}\nabla_x \times \overline{\mathbf{H}}}\\ \overline{-\mathrm{i}\nabla_x \times \overline{\mathbf{E}}}\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\nabla_x \times \mathbf{H}\\ +\mathrm{i}\nabla_x \times \mathbf{E}\end{pmatrix} \\ &\overset{[1]}{=} -\mathbf{Rot}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}. \end{aligned}$$

(iv) First, let us conjugate $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ with C:

$$C\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,C \overset{[1]}{=} \mathrm{e}^{+\mathrm{i}t\,C\,\mathbf{Rot}\,C} \overset{[1]}{=} \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$$

Thus, using $C^2 = 1$, this also implies that $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ commutes with C:

$$\left[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, C\right] \overset{[1]}{=} \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,C - C\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} \overset{[1]}{=} \left(\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} - C\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,C\right)C \overset{[1]}{=} 0$$

(v) Since the identity commutes with anything, the result follows directly from (iv):

$$\left[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, \mathrm{Re}\right] \overset{[1]}{=} \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, 1\right] + \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, C\right] = 0 + 0 \overset{[1]}{=} 0$$

(vi) $\left(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\right)$ is real-valued if and only if $\left(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\right) = \mathrm{Re}\left(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\right)$, and hence

$$\begin{aligned} \left(\mathbf{E}(t), \mathbf{H}(t)\right) &\overset{[1]}{=} \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\left(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\right) = \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,\mathrm{Re}\left(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\right) \overset{[1]}{=} \mathrm{Re}\;\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\left(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\right) \\ &\overset{[1]}{=} \mathrm{Re}\left(\mathbf{E}(t), \mathbf{H}(t)\right) \end{aligned}$$

has to be real.

24. Multiplication operators (23 points)

Let $V \in L^\infty(\mathbb{R}^n)$ and for $1 \leq p < \infty$ define the multiplication operator

$$(T_V \psi)(x) := V(x)\, \psi(x)\,, \qquad\qquad \psi \in L^p(\mathbb{R}^n)\,.$$

(i) Show that $T_V : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is bounded.

(ii) Prove that $\|T_V\| = \|V\|_\infty$ where $\|\cdot\|$ is the operator norm and $\|\cdot\|_\infty$ the L^∞-norm.

(iii) Show that a multiplication operator T_V is bounded if and only if $V \in L^\infty(\mathbb{R}^n)$.

(iv) Assume $V \in L^\infty(\mathbb{R}^n)$ is real-valued. Show that then $\langle \varphi, T_V \psi \rangle_{L^2(\mathbb{R}^n)} = \langle T_V \varphi, \psi \rangle_{L^2(\mathbb{R}^n)}$ holds for all $\varphi, \psi \in L^2(\mathbb{R}^n)$.

(v) Assume that V is bounded away from 0 and $+\infty$, i. e. that there exist $c, C > 0$ so that

$$0 < c \leq V(x) \leq C < +\infty$$

holds for all $x \in \mathbb{R}^n$. Show that T_V is invertible with bounded inverse.

Solution:

(i) From the elementary estimate $\big|(T_V\psi)(x)\big| = \big|V(x)\,\psi(x)\big| \leq \|V\|_\infty\ \big|\psi(x)\big|$ [1], we deduce

$$\begin{aligned}
\big\|T_V \psi\big\|_p &\overset{[1]}{=} \left(\int_{\mathbb{R}^n} \mathrm{d}x\, \big|(T_V\psi)(x)\big|^p \right)^{1/p} \\
&\overset{[1]}{\leq} \left(\int_{\mathbb{R}^n} \mathrm{d}x\, \|V\|_\infty^p |\psi(x)|^p \right)^{1/p} \\
&= \|V\|_\infty \left(\int_{\mathbb{R}^n} \mathrm{d}x\, |\psi(x)|^p \right)^{1/p} \\
&\overset{[1]}{=} \|V\|_\infty\, \|\psi\|_p\,.
\end{aligned}$$

Hence, T_V is bounded [1].

(ii) In (i), we have already shown $\|T_V\| \leq \|V\|_\infty$ [1] and it remains to show $\|T_V\| \geq \|V\|_\infty$. To do that, we will construct a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset L^p(\mathbb{R}^n)$ of normalized vectors so that

$$\lim_{j\to\infty} \big\|T_V \psi_j\big\| = \|V\|_\infty\ . \qquad [1]$$

[Any sequence of vectors gives 4 points in total.] For instance, one can use the following sequence of normalized step functions: let $U_j \subset |V|^{-1}\big(\big(\|V\|_\infty - 1/j, +\infty\big)\big)$ be a subset of non-zero measure and finite. The fact that such a set exists follows from the definition of the essential supremum which implies $|V|^{-1}\big(\big(\|V\|_\infty - 1/j, +\infty\big)\big)$ always has positive measure. The sequence is now defined in terms of the indicator function

$$1_{U_j}(x) := \begin{cases} 1 & x \in U_j \\ 0 & x \notin U_j \end{cases}\,.$$

Suitably normalized, we obtain our sequence,

$$\psi_j(x) := \frac{1_{U_j}(x)}{\big\|1_{U_j}\big\|_p}\,,$$

and by definition, we deduce

$$\big|(T_V \psi_j)(x)\big| \geq \big|\|V\|_\infty - 1/j\big| \, |\psi_j(x)|$$

which implies

$$\begin{aligned} \big\|T_V \psi_j\big\|_p &\geq \big|\|V\|_\infty - 1/j\big| \, \|\psi_j\|_p = \big|\|V\|_\infty - 1/j\big| \\ &\xrightarrow{j \to \infty} \|V\|_\infty \,. \end{aligned}$$

This shows $\big\|T_V\big\| = \|V\|_\infty$.

(iii) Our arguments in (i) have shown that $V \in L^\infty(\mathbb{R}^n)$ implies T_V is bounded [1].

Now suppose a multiplication operator T_V is bounded, but that $V \not\in L^\infty(\mathbb{R}^n)$ [1]. Since V is not bounded, there exists a sequence of vectors $\{\psi_j\}_{j \in \mathbb{N}} \subset L^p(\mathbb{R}^n)$ so that $\big|(T_V \psi_j)(x)\big| \geq j \, \big|\psi_j(x)\big|$ (e. g. modify the sequence constructed in (ii) appropriately) [1], and hence the norm

$$\big\|T_V \psi_j\big\|_p \geq j \, \|\psi_j\|_p \xrightarrow{j \to \infty} +\infty$$

explodes as $j \to \infty$ [1]. Hence, T_V cannot be bounded, contradiction! [1]

(iv) The claim follows from $\overline{V} = V$ and direct computation: for any $\varphi, \psi \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} \langle \varphi, T_V \psi \rangle_{L^2(\mathbb{R}^n)} &\overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}x \, \overline{\varphi(x)} \, (T_V \psi)(x) \overset{[1]}{=} \int_{\mathbb{R}^n} \mathrm{d}x \, \overline{\varphi(x)} \, V(x) \, \psi(x) \\ &= \int_{\mathbb{R}^n} \mathrm{d}x \, \overline{V(x) \, \varphi(x)} \, \psi(x) = \int_{\mathbb{R}^n} \mathrm{d}x \, \overline{(T_V \varphi)(x)} \, \psi(x) \\ &\overset{[1]}{=} \langle T_V \varphi, \psi \rangle_{L^2(\mathbb{R}^n)} \,. \end{aligned}$$

(v) Since V is bounded away from 0 and $+\infty$, so is V^{-1} [1],

$$0 < C^{-1} \leq V^{-1}(x) \leq c^{-1} < \infty \,.$$

Hence, also $T_{V^{-1}} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is a bounded multiplication operator by (i) [1]. Moreover, by direct computation, we verify that $T_{V^{-1}}$ is the inverse to T_V [1], e. g.

$$\begin{aligned} \big(T_V \, T_{V^{-1}} \psi\big)(x) &= V(x) \, \big(T_{V^{-1}} \psi\big)(x) \\ &= V(x) \, V^{-1}(x) \, \psi(x) = \psi(x) \,, \end{aligned}$$

and similarly $T_{V^{-1}} \, T_V = \mathrm{id}_{L^p(\mathbb{R}^n)}$ [1].

25. Boundedness of linear operators (8 points)

Find out whether the following operators are bounded or unbounded. Justify your answer!

(i) $H = -\partial_x^2$ on $L^2([-\pi,+\pi])$ with Dirichlet boundary conditions

(ii) $\mathrm{e}^{+\mathrm{i}t\partial_x^2}$ on $L^2([-\pi,+\pi])$ with Dirichlet boundary conditions

(iii) The multiplication operator associated to $V(x) = \frac{1}{|x|}$ on $L^2(\mathbb{R}^3)$

(iv) The multiplication operator associated to $V(x) = x^2$ on $L^2([-\pi,+\pi])$

Solution:

(i) By the arguments in Chapter 4.2.5, any $\psi \in L^2([-\pi,+\pi])$ can be expressed in terms of the orthonormal basis $\{\mathrm{e}^{+inx}\}_{n\in\mathbb{Z}}$,

$$\psi(x) = \sum_{n\in\mathbb{Z}} \widehat{\psi}(n)\, \mathrm{e}^{+inx} ,$$

where $\bigl\{\widehat{\psi}(n)\bigr\}_{n\in\mathbb{Z}}$ is a square summable sequence. Then formally, we compute

$$-\bigl(\partial_x^2\psi\bigr)(x) = \sum_{n\in\mathbb{Z}} n^2\, \widehat{\psi}(n)\, \mathrm{e}^{+inx} .$$

Since $\bigl\{n^2\, \widehat{\psi}(n)\bigr\}_{n\in\mathbb{Z}}$ need not be square summable (it need not even be a sequence converging to 0), $-\partial_x^2\psi$ need not exist in $L^2([-\pi,+\pi])$ [1]. Hence, $-\partial_x^2$ is *unbounded* [1].

(ii) By the arguments in Chapter 4.2.5, $\mathrm{e}^{+\mathrm{i}t\partial_x^2}$ is *bounded* [1], because $\bigl\|\mathrm{e}^{+\mathrm{i}t\partial_x^2}\psi\bigr\| = \|\psi\|$ holds for all $\psi \in L^2([-\pi,+\pi])$ according to the calculation outlined there [1].

(iii) $V(x) = \frac{1}{|x|}$ is *unbounded*, and hence, by problem 24 (iii) [1], the associated multiplication operator is also *unbounded* [1].

(iv) This operator is *bounded* by π^2 [1], because

$$\begin{aligned}\bigl\|T_{x^2}\psi\bigr\|^2 &= \int_{-\pi}^{+\pi} \mathrm{d}x\, \bigl|x^2\,\psi(x)\bigr|^2 \leq \pi^4 \int_{-\pi}^{+\pi} \mathrm{d}x\, \bigl|\psi(x)\bigr|^2 \\ &= \Bigl(\pi^2\, \|\psi\|\Bigr)^2\end{aligned}$$

holds for all $\psi \in L^2([-\pi,+\pi])$ [1].