UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 7
(2013.10.24)

Weighted Hilbert spaces, the free Maxwell equations & Operators

Homework Problems

22. Weighted L^2-spaces (16 points)

Let $\varepsilon \in L^\infty(\mathbb{R}^n)$ be a function bounded away from 0 and $+\infty$, i. e. there exist $c, C > 0$ such that

$$0 < c \leq \varepsilon(x) \leq C < +\infty$$

holds for almost all $x \in \mathbb{R}^n$. Define the weighted L^2-space $L^2_\varepsilon(\mathbb{R}^n)$ as the pre-Hilbert space with scalar product

$$\langle f, g \rangle_\varepsilon := \int_{\mathbb{R}^n} \mathrm{d}x \, \varepsilon(x) \, \overline{f(x)} \, g(x) \tag{1}$$

so that $\lVert f \rVert_\varepsilon := \sqrt{\langle f, f \rangle_\varepsilon} < \infty$.

(i) Show that $f \in L^2(\mathbb{R}^n)$ if and only if $f \in L^2_\varepsilon(\mathbb{R}^n)$.

(ii) Show that the map

$$U_\varepsilon : L^2(\mathbb{R}^n) \longrightarrow L^2_\varepsilon(\mathbb{R}^n) \, , \; f \mapsto \sqrt{\varepsilon} f \, ,$$

is norm-preserving, i. e. $\lVert f \rVert_\varepsilon = \lVert U_\varepsilon f \rVert_{L^2(\mathbb{R}^n)}$ holds for all $f \in L^2(\mathbb{R}^n)$.

(iii) Show that $L^2_\varepsilon(\mathbb{R}^n)$ is indeed a Hilbert space, i. e. prove that it is complete.

23. The free Maxwell equations as Schrödinger-type equation (17 points)

Consider the dynamical Maxwell equations

$$\begin{aligned} \partial_t \mathbf{E}(t) &= +\nabla_x \times \mathbf{H}(t) \, , & \mathbf{E}(0) &= \mathbf{E}^{(0)} \in L^2(\mathbb{R}^3, \mathbb{C}^3) \, , \\ \partial_t \mathbf{H}(t) &= -\nabla_x \times \mathbf{E}(t) \, , & \mathbf{H}(0) &= \mathbf{H}^{(0)} \in L^2(\mathbb{R}^3, \mathbb{C}^3) \, . \end{aligned} \tag{2}$$

Here, $L^2(\mathbb{R}^n, \mathbb{C}^N)$ is the Hilbert space with scalar product

$$\langle \Psi, \Phi \rangle := \int_{\mathbb{R}^3} \mathrm{d}x \, \Psi(x) \cdot \Phi(x)$$

defined in terms of the scalar product $\Psi(x) \cdot \Phi(x) := \sum_{j=1}^N \overline{\Psi_j(x)} \, \Phi_j(x)$ on $\mathbb{C}^N$.

Moreover, consider also the Schrödinger-type equation

$$\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \mathbf{Rot} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} \, , \qquad \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \end{pmatrix} = \begin{pmatrix} \mathbf{E}^{(0)} \\ \mathbf{H}^{(0)} \end{pmatrix} \in L^2(\mathbb{R}^3, \mathbb{C}^6) \, , \tag{3}$$

where the free Maxwell operator

$$\mathbf{Rot} := \begin{pmatrix} 0 & +\mathrm{i}\nabla_x^{\times} \\ -\mathrm{i}\nabla_x^{\times} & 0 \end{pmatrix}$$

is defined in terms of the curl $\nabla_x^{\times}\mathbf{E} := \nabla_x \times \mathbf{E}$.

During the computations, you may work with Rot and $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ as if they were $n \times n$ matrices.

(i) Verify that $\big(\mathbf{E}(t), \mathbf{H}(t)\big) := \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} \big(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\big)$ solves (3).

(ii) Show that the dynamical Maxwell equations (2) can be recast in the form (3).

(iii) Define complex conjugation C as $(C\Psi)(x) := \overline{\Psi(x)}$. Confirm that $C\,\mathbf{Rot}\,C = -\mathbf{Rot}$ holds.

(iv) Prove $C\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,C = \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ as well as that $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ commutes with complex conjugation, i. e.

$$\big[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, C\big] := \mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}\,C - C\,\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}} = 0\,.$$

(v) Show that $\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}$ commutes with the real part operator $\mathrm{Re} := \frac{1}{2}\big(1 + C\big)$, i. e. $\big[\mathrm{e}^{-\mathrm{i}t\mathbf{Rot}}, \mathrm{Re}\big] = 0$.

(vi) Show that if $\big(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}\big)$ is initially real-valued, then the solution $\big(\mathbf{E}(t), \mathbf{H}(t)\big)$ to the Maxwell equations is also real-valued.

24. Multiplication operators (23 points)

Let $V \in L^{\infty}(\mathbb{R}^n)$ and for $1 \leq p < \infty$ define the multiplication operator

$$(T_V\psi)(x) := V(x)\,\psi(x)\,, \qquad \psi \in L^p(\mathbb{R}^n)\,.$$

(i) Show that $T_V : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is bounded.

(ii) Prove that $\lVert T_V \rVert = \lVert V \rVert_{\infty}$ where $\lVert\cdot\rVert$ is the operator norm and $\lVert\cdot\rVert_{\infty}$ the L^{∞}-norm.

(iii) Show that a multiplication operator T_V is bounded if and only if $V \in L^{\infty}(\mathbb{R}^n)$.

(iv) Assume $V \in L^{\infty}(\mathbb{R}^n)$ is real-valued. Show that then $\langle \varphi, T_V\psi \rangle_{L^2(\mathbb{R}^n)} = \langle T_V\varphi, \psi \rangle_{L^2(\mathbb{R}^n)}$ holds for all $\varphi, \psi \in L^2(\mathbb{R}^n)$.

(v) Assume that V is bounded away from 0 and $+\infty$, i. e. that there exist $c, C > 0$ so that

$$0 < c \leq V(x) \leq C < +\infty$$

holds for all $x \in \mathbb{R}^n$. Show that T_V is invertible with bounded inverse.

25. Boundedness of linear operators (8 points)

Find out whether the following operators are bounded or unbounded. Justify your answer!

(i) $H = -\partial_x^2$ on $L^2([-\pi, +\pi])$ with Dirichlet boundary conditions

(ii) $\mathrm{e}^{+\mathrm{i}t\partial_x^2}$ on $L^2([-\pi, +\pi])$ with Dirichlet boundary conditions

(iii) The multiplication operator associated to $V(x) = \frac{1}{|x|}$ on $L^2(\mathbb{R}^3)$

(iv) The multiplication operator associated to $V(x) = x^2$ on $L^2([-\pi, +\pi])$

Hand in home work on: Thursday, 31 October 2013, before class