UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Solutions 8
(2014.11.04)

Time-reversal and von Neumann's Theorem

Homework Problems

24. Time-reversal symmetry (16 points)

Let $(C\psi)(x) := \overline{\psi(x)}$ be complex conjugation defined on $L^2(\mathbb{R}^3)$.

(i) Show that C is a *conjugation*, i. e. an antiunitary ($\langle\varphi,\psi\rangle = \overline{\langle C\varphi, C\psi\rangle} = \langle C\psi, C\varphi\rangle$ for all $\varphi,\psi \in L^2(\mathbb{R}^d)$) which squares to $\mathrm{id}_{L^2(\mathbb{R}^3)}$.

Now consider the magnetic Schrödinger operator

$$H^A = \bigl(-\mathrm{i}\nabla_x - A(\hat{x})\bigr)^2 + V(\hat{x})$$

with domain $\mathcal{D}(H^A) = \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$ where the magnetic vector potential $A \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$ is associated to the magnetic field $B = \nabla_x \times A$, and the real-valued potential $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ satisfies the conditions of Theorem 5.2.24.

(ii) Show $C\,H^A\,C = H^{-A}$.

(iii) Let $H := H^{A=0}$ be the non-magnetic Schrödinger operator. Prove $[H, C] = 0$.

(iv) Show that C implements physical time-reversal for H from part (iii), i. e. $C\,U(t)\,C = U(-t)$ where $U(t) = \mathrm{e}^{-\mathrm{i}tH}$ is the time evolution group.

Solution:

(i) Evidently, C is antilinear [1], $C(\varphi + \mu\psi) = C\varphi + \bar{\mu}\,C\psi$, and an involution as $(C^2\psi)(x) = \overline{\overline{\psi(x)}} = \psi(x)$ [1]. The antiunitarity is quickly verified by hand, too:

$$\langle\varphi,\psi\rangle = \int_{\mathbb{R}^3} \mathrm{d}x\, \overline{\varphi(x)}\,\psi(x) \overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x\,(C\varphi)(x)\,\overline{(C\psi)(x)} \overset{[1]}{=} \langle C\psi, C\varphi\rangle$$

Hence, C is an antiunitary.

(ii) First of all, the domain $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3) = C\,\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$ [1] is left invariant under complex conjugation. For multiplication operators for real-valued functions such as $V(\hat{x})$ and $A_j(\hat{x})$, we deduce $C\,V(\hat{x})\,C = V(\hat{x})$ [1]. Moreover, $C\,(-\mathrm{i}\nabla_x)\,C = +\mathrm{i}\nabla_x$ holds true [1], and hence

$$C\,\bigl(-\mathrm{i}\partial_{x_j} - A_j(\hat{x})\bigr)\,C \overset{[1]}{=} -\bigl(-\mathrm{i}\partial_{x_j} + A_j(\hat{x})\bigr).$$

Put together, we obtain

$$\begin{aligned} C\,H^A\,C &\overset{[1]}{=} \sum_{j=1}^{3} C\,\bigl(-\mathrm{i}\partial_{x_j} - A_j(\hat{x})\bigr)\,C^2\,\bigl(-\mathrm{i}\partial_{x_j} - A_j(\hat{x})\bigr)\,C + C\,V(\hat{x})\,C \\ &= \sum_{j=1}^{3} (-1)^2\,\bigl(-\mathrm{i}\partial_{x_j} + A_j(\hat{x})\bigr)^2 + V(\hat{x}) \overset{[1]}{=} H^{-A}. \end{aligned}$$

(iii) This follows immediately from (ii) since for $A = 0$ the operators $H^{A=0}$ and $H^{-A=0} = C\,H^{A=0}\,C$ coincide [1], and thus, multiplying both sides with C from the left yields

$$C\,H = C^2\,H\,C \overset{[1]}{=} H\,C.$$

(iv) Multiplying the Schrödinger equation with C yields

$$C\big(\mathrm{i}\partial_t\psi\big) \overset{[1]}{=} -\mathrm{i}\partial_t C\psi = C\,H\psi \overset{[1]}{=} H\,C\psi.$$

Put another way, $C\,\mathrm{e}^{-\mathrm{i}tH}\psi = \mathrm{e}^{+\mathrm{i}tH}\,C\psi$ [1] or

$$C\,\mathrm{e}^{-\mathrm{i}tH}\,C \overset{[1]}{=} \mathrm{e}^{+\mathrm{i}tH}.$$

25. Von Neumann's Theorem (17 points)

(i) Prove the following theorem due to von Neumann:

Theorem 1 (von Neumann) *Let $H : \mathcal{D}(H) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ be a densely defined, symmetric operator on a Hilbert space. If there exists an antiunitary operator C with*

(a) $C^2 = \mathrm{id}_{\mathcal{H}}$,

(b) $C\mathcal{D}(H) \subseteq \mathcal{D}(H)$, *and*

(c) $[H, C] = 0$ *on* $\mathcal{D}(H)$,

then the deficiency indices agree, $N_+ = N_-$.

(ii) Assume $H = -\Delta_x + V$ with domain $\mathcal{D}(H) = \mathcal{C}_c^\infty(\mathbb{R}^d)$ is symmetric. Prove that then H always has a selfadjoint extension.

Hint: Review Chapter 5.2.1.

Solution:

(i) $C^2 = \mathrm{id}_{\mathcal{H}}$ actually implies $C\mathcal{D}(H) = \mathcal{D}(H)$ (equality) [1], because applying C to both sides of $C\mathcal{D}(H) \subseteq \mathcal{D}(H)$ yields $\mathcal{D}(H) \subseteq C\mathcal{D}(H)$ [1].

Now let $K_\pm := \ker\big(H^* \pm \mathrm{i}\big)$. We will show that $CK_\pm = K_\mp$ [1]: Let $\varphi_+ \in K_+$. Then we have for all $\psi \in \mathcal{D}(H)$

$$\begin{aligned} 0 &\overset{[1]}{=} \overline{\big\langle (H^* + \mathrm{i})\varphi_+, \psi \big\rangle} \overset{[1]}{=} \overline{\big\langle \varphi_+, (H - \mathrm{i})\psi \big\rangle} \\ &= \big\langle C\varphi_+, C(H - \mathrm{i})\psi \big\rangle \overset{[1]}{=} \big\langle C\varphi_+, (H + \mathrm{i})C\psi \big\rangle, \end{aligned}$$

or, put another way, $C\varphi_+ \in \mathrm{ran}\,(H + \mathrm{i})^\perp$ [1]. By Lemma 5.2.5 (i) this means $C\varphi_+ \in \ker(H^* - \mathrm{i}) = K_-$ [1], and thus, $CK_+ \subseteq K_-$ [1].

Repeating the same argument for $\varphi_- \in K_-$ yields also the opposite inclusion $CK_- \subseteq K_+$ [1]. Combining the two inclusions with $C^2 = \mathrm{id}_{\mathcal{H}}$ yields $K_- \subseteq CK_+ \subseteq K_-$, i. e. $CK_+ = K_-$ [1], and similarly also $CK_- = K_+$. Given that C is an isometry, that means the dimensions of K_- and K_+ – the deficiency indices N_- and N_+ – have to be the same, $N_- = N_+$ [1].

(ii) Here, we can pick complex conjugation C [1]. Evidently, $C^2 = \mathrm{id}_{\mathcal{H}}$, $C\mathcal{D}(H) = \mathcal{D}(H)$ [1], and by problem 24, we also have $[H, C] = 0$ [1]. Hence, the deficiency indices agree, $N_+ = N_-$ [1], and Theorem 5.2.7 applies, i. e. there exists *a* selfadjoint extension of H [1].