UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

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Solutions 8
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Operators

Homework Problems

26. Convergence of operators

Consider the following sequences $\{T_n\}_{n\in\mathbb{N}}$ of operators on the Hilbert space

$$\ell^2(\mathbb{N}) = \Big\{ a \equiv (a_n)_{n\in\mathbb{N}} \ \Big|\ \textstyle\sum_{n=1}^{\infty} |a_n|^2 < \infty \Big\}$$

and investigate whether they converge in norm, strongly or weakly:

(i) $T_n(a) := \big(\tfrac{1}{n}a_1, \tfrac{1}{n}a_2, \ldots\big)$

(ii) $T_n(a) := \big(\underbrace{0,\ldots,0}_{n \text{ places}}, a_{n+1}, a_{n+2}, \ldots\big)$

(iii) $T_n(a) := \big(\underbrace{0,\ldots,0}_{n \text{ places}}, a_1, a_2, \ldots\big)$

Solution:

(i) The sequence T_n converges in norm/uniformly to $0 \in \mathcal{B}\big(\ell^2(\mathbb{N})\big)$, because

$$\big\|T_n(a)\big\|_{\ell^2(\mathbb{N})} = \tfrac{1}{n}\,\|a\|_{\ell^2(\mathbb{N})} \xrightarrow{n\to\infty} 0$$

and thus $\big\|T_n\big\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = 1/n$. The above equation also implies that T_n converges to 0 also strongly and weakly, because

$$\big|\big\langle a, T_n(b)\big\rangle_{\ell^2(\mathbb{N})}\big| \le \big\|a\big\|_{\ell^2(\mathbb{N})}\,\big\|T_n(b)\big\|_{\ell^2(\mathbb{N})} \xrightarrow{n\to\infty} 0\,.$$

(ii) For a fixed $a \in \ell^2(\mathbb{N})$, we have

$$\big\|T_n(a)\big\|_{\ell^2(\mathbb{N})} = \sum_{j=1}^{\infty}\big|\big(T_n(a)\big)_j\big|^2 = \sum_{j=n+1}^{\infty}|a_j| \xrightarrow{n\to\infty} 0\,,$$

and thus T_n converges strongly (and weakly) to $0 \in \mathcal{B}\big(\ell^2(\mathbb{N})\big)$. However, if $e_n := (\delta_{jn})_{j\in\mathbb{N}} = \big(0,\ldots,0,1,0,\ldots\big)$, we see that

$$\big\|T_n e_{n+1}\big\|_{\ell^2(\mathbb{N})} = 1\,,$$

and thus T_n does not converge to 0 in norm, because $\|T_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \ge 1$.

(iii) T_n converges weakly to 0:

$$\begin{aligned}\left|\left\langle a, T_n(b)\right\rangle_{\ell^2(\mathbb{N})}\right| &= \left|\sum_{j=1}^{\infty} \overline{b_j}\left(T_n(a)\right)_j\right| = \left|\sum_{j=n+1}^{\infty} \overline{b_j}\, a_{j-n}\right| \\ &\leq \left(\sum_{j=n+1}^{\infty} |b_j|^2\right)^{1/2} \left(\sum_{j=n+1}^{\infty} \left|a_{j-n}\right|^2\right)^{1/2} = \left(\sum_{j=n+1}^{\infty} |b_j|^2\right)^{1/2} \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{1/2} \\ &\xrightarrow{n\to\infty} 0\end{aligned}$$

However, it does not converge strongly or in norm, because

$$\left\|T_n(a)\right\|_{\ell^2(\mathbb{N})}^2 = \sum_{j=n+1}^{\infty} \left|a_{j-n}\right|^2 = \sum_{j=1}^{\infty} |a_j|^2 = \|a\|_{\ell^2(\mathbb{N})}\,.$$

27. Symmetric operators (17 points)

Let $H = \frac{1}{2m}(-\mathrm{i}\nabla_x)^2 + V$ be a Hamilton operator with potential $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$.

Define the smooth functions with compact support as

$$\mathcal{C}_c^\infty(\mathbb{R}^3) := \big\{\varphi : \mathbb{R}^3 \longrightarrow \mathbb{C} \mid \varphi \in \mathcal{C}^\infty(\mathbb{R}^3),\ \operatorname{supp}\varphi \text{ compact}\big\}.$$

(i) Prove $\mathcal{C}_c^\infty(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$.

(ii) Show that H is *symmetric* on $\mathcal{C}_c^\infty(\mathbb{R}^3)$, i. e. that

$$\big\langle \varphi, H\psi \big\rangle = \big\langle H\varphi, \psi \big\rangle$$

holds for all $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$.

Solution:

(i) Every smooth function with compact support is square-integrable: let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, then there exists a compact subset $K \subset \mathbb{R}^3$, so that

$$\operatorname{supp}\varphi = \overline{\big\{x \in \mathbb{R}^3 \mid \varphi(x) \neq 0\big\}} \subseteq K\,. \qquad [1]$$

Since φ is also continuous, we can estimate the supremum from above by

$$\|\varphi\|_\infty = \sup_{x\in\mathbb{R}^3} |\varphi(x)| = \sup_{x\in K} |\varphi(x)| < \infty\,. \qquad [1]$$

Hence, we obtain

$$\|\varphi\|^2 = \int_{\mathbb{R}^3} \mathrm{d}x\ |\varphi(x)|^2 \overset{[1]}{=} \int_K \mathrm{d}x\ |\varphi(x)|^2 \overset{[1]}{\leq} |K|\ \big(\sup_{x\in K} |\varphi(x)|\big)^2 < \infty.$$

(ii) We will treat kinetic and potential energy separately: clearly, derivatives map $\mathcal{C}_c^\infty(\mathbb{R}^3)$ into itself, and thus $(-\mathrm{i}\nabla_x)^2\varphi \in L^2(\mathbb{R}^3)$ [1]. Fix $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$. Then there exists a compact set $K \subset \mathbb{R}^3$ whose *interior* contains $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ [1]. Then we compute using repeated partial integration

$$\begin{aligned}
\big\langle \varphi, \tfrac{1}{2m}(-\mathrm{i}\nabla_x)^2\psi \big\rangle &= \sum_{j=1}^3 \frac{1}{2m} \big\langle \varphi, (-\mathrm{i}\partial_{x_j})^2\psi \big\rangle \overset{[1]}{=} \sum_{j=1}^3 \frac{1}{2m} \int_{\mathbb{R}^3} \mathrm{d}x\, \overline{\varphi(x)}\, \big((-\mathrm{i}\partial_{x_j})^2\psi\big)(x) \\
&\overset{[1]}{=} \sum_{j=1}^3 \frac{1}{2m} \int_K \mathrm{d}x\, \overline{\varphi(x)}\, \big((-\mathrm{i}\partial_{x_j})^2\psi\big)(x) \\
&\overset{[1]}{=} \sum_{j=1}^3 \frac{1}{2m} \int_{\partial K} \mathrm{d}S(x)\, \overline{\varphi(x)}\, \big((-\mathrm{i})^2\partial_{x_j}\psi\big)(x) + \\
&\qquad - \sum_{j=1}^3 \frac{1}{2m} \int_K \mathrm{d}x\, \overline{\partial_{x_j}\varphi(x)}\, \big((-\mathrm{i})^2\partial_{x_j}\psi\big)(x) \\
&\overset{[1]}{=} 0 - \sum_{j=1}^3 \frac{1}{2m} (-\mathrm{i})^2 \int_{\partial K} \mathrm{d}S(x)\, \overline{\partial_{x_j}\varphi(x)}\, \psi(x) + \\
&\qquad + \sum_{j=1}^3 \frac{1}{2m} (-\mathrm{i})^2 \int_K \mathrm{d}x\, \overline{\partial_{x_j}^2\varphi(x)}\, \psi(x)
\end{aligned}$$

$$\overset{[1]}{=} \sum_{j=1}^{3} \frac{1}{2m} \int_K \mathrm{d}x \, \overline{\big((-\mathrm{i}\partial_{x_j})^2 \varphi\big)(x)} \, \psi(x)$$

$$\overset{[1]}{=} \big\langle \tfrac{1}{2m} (-\mathrm{i}\nabla_x)^2 \varphi , \psi \big\rangle$$

Here, $\mathrm{d}S(x)$ is the surface measure on ∂K. The boundary terms vanish, because φ and ψ as well as their derivatives vanish on ∂K.

Now to the potential energy: since V is continuous, it is bounded on compact subsets. Choose any $\varphi, \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$. Then $V\varphi \in L^2(\mathbb{R}^3)$ [1] and hence,

$$\big\langle \varphi , V \psi \big\rangle \overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\varphi(x)} \, \big(V\psi\big)(x) \overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\varphi(x)} \, V(x) \, \psi(x)$$

$$\overset{[1]}{=} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\big(V\varphi\big)(x)} \, \psi(x) \overset{[1]}{=} \big\langle V\varphi , \psi \big\rangle$$

holds.

28. Positive operators and the trace

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ and $\rho = \rho^*$ a *density operator*, i. e. $0 \leq \rho$ which in addition satisfies

$$\mathrm{Tr}\,\rho = \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n \rangle = 1.$$

(i) Show that the trace is independent of the choice of basis $\{\varphi_n\}_{n\in\mathbb{N}}$.

(ii) Show that any rank-1 projection $P = \langle \psi_*, \cdot \rangle\ \psi_*$, $\|\psi_*\| = 1$, is a density operator.

(iii) Show that $\rho^2 = \rho$ if and only if ρ is a rank-1 projection.

Solution:

(i) By assumption, the sum

$$\mathrm{Tr}\,\rho = \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n \rangle = 1$$

converges to 1, and the positivity of ρ implies it also converges *absolutely* to 1.

To show that the sum is independent of the choice of orthonormal basis, let $\{\psi_j\}_{j\in\mathbb{N}}$ be a second orthonormal basis. Then we can express any φ_n from the first orthonormal basis in terms of the ψ_j,

$$\varphi_n = \sum_{j\in\mathbb{N}} \langle \psi_j, \varphi_n \rangle\ \psi_j\ .$$

Plugged into the sum, we obtain

$$\begin{aligned} 1 = \mathrm{Tr}\,\rho = \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n \rangle &= \sum_{j,l,n\in\mathbb{N}} \Big\langle \langle \psi_j, \varphi_n \rangle\ \psi_j, \rho\ \langle \psi_l, \varphi_n \rangle\ \psi_l \Big\rangle \\ &= \sum_{j,l,n\in\mathbb{N}} \overline{\langle \psi_j, \varphi_n \rangle}\ \langle \psi_l, \varphi_n \rangle\ \langle \psi_j, \rho\,\psi_l \rangle = \sum_{j,l,n\in\mathbb{N}} \Big\langle \psi_l, \langle \varphi_n, \psi_j \rangle\ \varphi_n \Big\rangle\ \langle \psi_j, \rho\,\psi_l \rangle \\ &= \sum_{j,l\in\mathbb{N}} \langle \psi_l, \psi_j \rangle\ \langle \psi_j, \rho\,\psi_l \rangle = \sum_{j\in\mathbb{N}} \langle \psi_j, \rho\,\psi_j \rangle\ . \end{aligned}$$

(ii) First of all, $P = \langle \psi_*, \cdot \rangle\ \psi_*$ is selfadjoint, because for all $\varphi, \phi \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} \big\langle \varphi, P\phi \big\rangle &= \Big\langle \varphi, \langle \psi_*, \phi \rangle\ \psi_* \Big\rangle = \langle \psi_*, \phi \rangle\ \langle \varphi, \psi_* \rangle \\ &= \Big\langle \langle \psi_*, \varphi \rangle\, \psi_*, \phi \Big\rangle = \big\langle P\varphi, \phi \big\rangle\,. \end{aligned}$$

Moreover, $P \geq 0$ because $P^2 = P$, and thus

$$\langle \varphi, P\varphi \rangle = \big\langle \varphi, P^2\varphi \big\rangle = \langle P^*\varphi, P\varphi \rangle = \langle P\varphi, P\varphi \rangle \geq 0\,.$$

By (i), we can compute the trace in any orthonormal basis, so for instance we can pick $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\varphi_1 = \psi_*$, and in that basis only one term of the sum survives,

$$\begin{aligned} \mathrm{Tr}\,P &= \sum_{n=1}^{\infty} \langle \varphi_n, P\varphi_n \rangle = \langle \psi_*, P\psi_* \rangle + \sum_{n=2}^{\infty} \langle \varphi_n, P\varphi_n \rangle \\ &= \langle \psi_*, \psi_* \rangle + 0 = 1\,. \end{aligned}$$

Thus, P is a density operator.

(iii) "⇐:" if ρ is a rank-1 projection, then $\rho^2 = \rho$ is a density operator by (ii).

"⇒:" Assume $\rho^2 = \rho$, i. e. ρ is an orthogonal projection (selfadjointness is included in the definition of ρ). Hence, we can spit $L^2(\mathbb{R}^n) = \operatorname{ran}\rho \oplus \big(\operatorname{ran}\rho\big)^{\perp}$ into the range of ρ and its orthogonal complement, and the action of ρ and $\psi = \psi_\rho + \psi_\rho^\perp$ is

$$\rho\psi = \rho\big(\psi_\rho + \psi_\rho^\perp\big) = \psi_\rho \, .$$

Thus, choosing a basis $\{\varphi_n\}_{n\in\mathbb{N}} = \{\varphi_n\}_{n\in\mathcal{I}} \cup \{\varphi_n\}_{n\in\mathbb{N}\setminus\mathcal{I}}$ where $\{\varphi_n\}_{n\in\mathcal{I}}$ is an orthonormal basis of $\operatorname{ran}\rho$, we compute

$$\begin{aligned} \operatorname{Tr}\rho &= \sum_{n\in\mathbb{N}} \langle\varphi_n, \rho\varphi_n\rangle = \sum_{n\in\mathcal{I}} \langle\varphi_n, \rho\varphi_n\rangle \\ &= \sum_{n\in\mathcal{I}} \langle\varphi_n, \varphi_n\rangle = |\mathcal{I}| \overset{!}{=} 1 \, . \end{aligned}$$

Since $|\mathcal{I}|$ is the dimensionality of $\operatorname{ran}\rho$, we deduce that $\dim\big(\operatorname{ran}\rho\big) = 1$, and thus, P is a rank-1 projection.

29. Extensions of operators

Consider the vector space $\mathrm{Pol}([0,1])$ of polynomials of finite degree with complex coefficients (seen as functions from $[0,1]$ to $\mathbb{C}$) and define the operator

$$p(x) = \sum_{n=0}^{N} \alpha_n \, x^n \longmapsto \mathrm{d}p(x) := \sum_{n=0}^{N} n \, \alpha_n \, x^{n-1}$$

on $\mathrm{Pol}([0,1])$.

(i) Consider the Banach space $\big(\mathcal{C}([0,1]), \lVert\cdot\rVert_0\big)$, $\lVert f \rVert_0 := \sup_{x\in[0,1]} \lvert f(x) \rvert$. Investigate whether d has a continuous extension $\widetilde{\mathrm{d}} : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$.

(ii) Consider the Banach space $\big(\mathcal{C}^1([0,1]), \lVert\cdot\rVert_1\big)$,

$$\lVert f \rVert_1 := \sup_{x\in[0,1]} \lvert f(x) \rvert + \sup_{x\in[0,1]} \lvert f'(x) \rvert \, .$$

Investigate whether d has a continuous extension to $\widetilde{\mathrm{d}} : \mathcal{C}^1([0,1]) \longrightarrow \mathcal{C}([0,1])$.

Hint: You may use without proof that $\mathrm{Pol}([0,1])$ is dense in $\mathcal{C}^k([0,1])$, $k = 0, 1$.

Solution:

(i) We have to check whether d is bounded: Clearly, the monomials $\{x^n\}_{n\in\mathbb{N}_0}$ are a basis of $\mathrm{Pol}([0,1])$ with $\lVert x^n \rVert_0 = 1$, and we see that

$$\lVert \mathrm{d} \rVert_{\mathrm{Pol}([0,1])} \geq \frac{\lVert \mathrm{d}(x^n) \rVert}{\lVert x^n \rVert} = \lVert n \, x^{n-1} \rVert = n \xrightarrow{n\to\infty} +\infty \, .$$

That means there cannot be any continuous extension.

(ii) The norm is now bounded,

$$\begin{aligned} \lVert \mathrm{d} \rVert_{\mathrm{Pol}([0,1])} &= \sup_{p\in\mathrm{Pol}([0,1])\setminus\{0\}} \frac{\lVert \mathrm{d}p \rVert_0}{\lVert p \rVert_1} = \sup_{p\in\mathrm{Pol}([0,1])\setminus\{0\}} \frac{\lVert p' \rVert_0}{\lVert p \rVert_0 + \lVert p' \rVert_0} \\ &\leq \sup_{p\in\mathrm{Pol}([0,1])\setminus\{0\}} \frac{\lVert p' \rVert_0}{\lVert p' \rVert_0} = 1 \, , \end{aligned}$$

so that by Theorem 5.1.6, there exists a bounded extension

$$\widetilde{\mathrm{d}} : \mathcal{C}^1([0,1]) \longrightarrow \mathcal{C}([0,1]) \, .$$