UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Problem Sheet 8
(2014.11.04)

Time-reversal and von Neumann's Theorem

Homework Problems

24. Time-reversal symmetry

Let $(C\psi)(x) := \overline{\psi(x)}$ be complex conjugation defined on $L^2(\mathbb{R}^3)$.

(i) Show that C is a *conjugation*, i. e. an antiunitary ($\langle\varphi,\psi\rangle = \overline{\langle C\varphi, C\psi\rangle} = \langle C\psi, C\varphi\rangle$ for all $\varphi,\psi \in L^2(\mathbb{R}^d)$) which squares to $\mathrm{id}_{L^2(\mathbb{R}^3)}$.

Now consider the magnetic Schrödinger operator

$$H^A = \big(-\mathrm{i}\nabla_x - A(\hat{x})\big)^2 + V(\hat{x})$$

with domain $\mathcal{D}(H^A) = \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$ where the magnetic vector potential $A \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$ is associated to the magnetic field $B = \nabla_x \times A$, and the real-valued potential $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ satisfies the conditions of Theorem 5.2.24.

(ii) Show $C\,H^A\,C = H^{-A}$.

(iii) Let $H := H^{A=0}$ be the non-magnetic Schrödinger operator. Prove $[H, C] = 0$.

(iv) Show that C implements physical time-reversal for H from part (iii), i. e. $C\,U(t)\,C = U(-t)$ where $U(t) = \mathrm{e}^{-\mathrm{i}tH}$ is the time evolution group.

25. Von Neumann's Theorem

(i) Prove the following theorem due to von Neumann:

Theorem 1 (von Neumann) *Let $H : \mathcal{D}(H) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ be a densely defined, symmetric operator on a Hilbert space. If there exists an antiunitary operator C with*

(a) $C^2 = \mathrm{id}_{\mathcal{H}}$,

(b) $C\mathcal{D}(H) \subseteq \mathcal{D}(H)$, and

(c) $[H, C] = 0$ on $\mathcal{D}(H)$,

then the deficiency indices agree, $N_+ = N_-$.

(ii) Assume $H = -\Delta_x + V$ with domain $\mathcal{D}(H) = \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ is symmetric. Prove that then H always has a selfadjoint extension.

Hint: Review Chapter 5.2.1.

Hand in home work on: Friday, 7 November 2014, before class