UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 8
(2013.10.31)

Operators

Homework Problems

26. Convergence of operators

Consider the following sequences $\{T_n\}_{n\in\mathbb{N}}$ of operators on the Hilbert space

$$\ell^2(\mathbb{N}) = \Big\{ a \equiv (a_n)_{n\in\mathbb{N}} \;\big|\; \textstyle\sum_{n=1}^{\infty} |a_n|^2 < \infty \Big\}$$

and investigate whether they converge in norm, strongly or weakly:

(i) $T_n(a) := \big(\tfrac{1}{n} a_1, \tfrac{1}{n} a_2, \ldots\big)$

(ii) $T_n(a) := \big(\underbrace{0, \ldots, 0}_{n \text{ places}}, a_{n+1}, a_{n+2}, \ldots\big)$

(iii) $T_n(a) := \big(\underbrace{0, \ldots, 0}_{n \text{ places}}, a_1, a_2, \ldots\big)$

27. Symmetric operators

Let $H = \frac{1}{2m}(-\mathrm{i}\nabla_x)^2 + V$ be a Hamilton operator with potential $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$.
Define the smooth functions with compact support as

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3) := \big\{ \varphi : \mathbb{R}^3 \longrightarrow \mathbb{C} \mid \varphi \in \mathcal{C}^\infty(\mathbb{R}^3),\ \mathrm{supp}\,\varphi \text{ compact} \big\}.$$

(i) Prove $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$.

(ii) Show that H is *symmetric* on $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$, i. e. that

$$\big\langle \varphi, H\psi \big\rangle = \big\langle H\varphi, \psi \big\rangle$$

holds for all $\varphi, \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$.

28. Positive operators and the trace

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ and $\rho = \rho^*$ a *density operator,* i. e. $0 \leq \rho$ which in addition satisfies

$$\mathrm{Tr}\,\rho = \sum_{n\in\mathbb{N}} \langle \varphi_n, \rho\varphi_n \rangle = 1.$$

(i) Show that the trace is independent of the choice of basis $\{\varphi_n\}_{n\in\mathbb{N}}$.

(ii) Show that any rank-1 projection $P = \langle \psi_*, \cdot \rangle\ \psi_*$, $\|\psi_*\| = 1$, is a density operator.

(iii) Show that $\rho^2 = \rho$ if and only if ρ is a rank-1 projection.

29. Extensions of operators

Consider the vector space Pol$([0,1])$ of polynomials of finite degree with complex coefficients (seen as functions from $[0,1]$ to $\mathbb{C}$) and define the operator

$$p(x) = \sum_{n=0}^{N} \alpha_n \, x^n \longmapsto \mathrm{d}p(x) := \sum_{n=0}^{N} n \, \alpha_n \, x^{n-1}$$

on Pol$([0,1])$.

(i) Consider the Banach space $\big(\mathcal{C}([0,1]), \|\cdot\|_0\big)$, $\|f\|_0 := \sup_{x\in[0,1]}|f(x)|$. Investigate whether d has a continuous extension $\widetilde{\mathrm{d}} : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$.

(ii) Consider the Banach space $\big(\mathcal{C}^1([0,1]), \|\cdot\|_1\big)$,

$$\|f\|_1 := \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| \, .$$

Investigate whether d has a continuous extension to $\widetilde{\mathrm{d}} : \mathcal{C}^1([0,1]) \longrightarrow \mathcal{C}([0,1])$.

Hint: You may use without proof that Pol$([0,1])$ is dense in $\mathcal{C}^k([0,1])$, $k = 0, 1$.

Remark: To decrease the work load of the TA, only *one* will be marked this week. It will be announced only after the due date which of the four problems will be marked. Starting from Sheet 09, *about two* problems will be marked. In any case, unmarked problems will be relevant for tests 2–3 and the final exam.

Hand in home work on: Thursday, 7 November 2013, before class