The discrete Fourier transform

Homework Problems

30. The Fourier transform of the sawtooth and the tent function

Consider the function $f(x) := \max\{0, x\}$ on $(-\pi, +\pi]$. Determine the Fourier coefficients, their asymptotic behavior for large $|k|$ of

(i) f,

(ii) g with $g(x) = f(-x)$ and

(iii) $h = f + g$.

In each of the cases, sketch the graph and give the first few terms of the sin and cos representation.

Hint: Work smart, not hard.

Solution:

(i) We compute directly for $k = 0$

$$(\mathcal{F}f)(0) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}x\, f(x) = \frac{1}{2\pi}\int_{0}^{+\pi} \mathrm{d}x\, x = \left[\frac{1}{4\pi}x^2\right]_0^\pi = \frac{\pi}{4}$$

and for $k \neq 0$

$$\begin{aligned}(\mathcal{F}f)(k) &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, f(x) = \frac{1}{2\pi}\int_{0}^{+\pi} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, x \\ &= \left[\frac{1}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}kx}}{-\mathrm{i}k}\, x\right]_0^\pi - \frac{1}{2\pi}\int_0^\pi \mathrm{d}x\, \frac{\mathrm{e}^{-\mathrm{i}kx}}{-\mathrm{i}k} \\ &= \frac{1}{2\pi}\left(\frac{(-1)^k\,\pi}{-\mathrm{i}k} + \frac{(-1)^k - 1}{(-\mathrm{i}k)^2}\right) = \frac{\mathrm{i}\,(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2}.\end{aligned}$$

Clearly, $\lim_{|k|\to\infty}\left|k\,\hat{f}(k)\right| = 1/2$, i. e. $\hat{f}(k) = \mathcal{O}(|k|^{-1})$.

Expressed in terms of sin and cos, we obtain

$$\begin{aligned}f(x) = \frac{\pi}{4} - \frac{2\cos x}{\pi} - \frac{2\cos 3x}{9\pi} - \ldots \\ - \sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} \pm \ldots\end{aligned}$$

(ii) The Fourier coefficients of $g(x) = f(-x)$ are

$$\hat{g}(k) = \hat{f}(-k) = \begin{cases} \frac{\pi}{4} & k = 0 \\ -\frac{\mathrm{i}\,(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2} & k \in \mathbb{Z} \setminus \{0\} \end{cases}$$

$$= \begin{cases} \frac{\pi}{4} & k = 0 \\ -\frac{\mathrm{i}}{2k} & k \in 2\mathbb{Z} \setminus \{0\} \\ +\frac{\mathrm{i}}{2k} - \frac{1}{\pi k^2} & k \in 2\mathbb{Z} + 1 \end{cases},$$

and hence we obtain again $\hat{g}(k) = \mathcal{O}(|k|^{-1})$. Similarly, the signs of the sin terms flip while those for the cos terms remain the same,

$$\begin{aligned} g(x) = \frac{\pi}{4} - \frac{2\cos x}{\pi} - \frac{2\cos 3x}{9\pi} - \dots \\ + \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \pm \dots \\ = f(-x). \end{aligned}$$

(iii) By linearity, the Fourier coefficients of $h = f + g$ are the sum of the Fourier coefficients for f and g,

$$\hat{h}(k) = \begin{cases} \frac{\pi}{2} & k = 0 \\ 0 & k \neq 0,\ k \in 2\mathbb{Z} \\ -\frac{2}{\pi k^2} & k \in 2\mathbb{Z} + 1 \end{cases},$$

and we immediately obtain the sin and cos expansion,

$$\begin{aligned} h(x) &= f(x) + g(x) = f(x) + f(-x) \\ &= \frac{\pi}{2} - \frac{4\cos x}{\pi} - \frac{4\cos 3x}{9\pi} - \dots . \end{aligned}$$

The coefficients now decay like $1/k^2$, $\hat{f}(k) = \mathcal{O}(1/k^2)$.

31. Fourier series of particular functions

(i) Which periodic function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the Fourier coefficients $\hat{f}(k) = \frac{1}{|k|!}$, $k \in \mathbb{Z}$?

(ii) What are the Fourier coefficients of $g(x) = \mathrm{e}^{\sin x} \cos\big(\cos x\big)$?

(iii) What are the Fourier coefficients of $h(x) = \mathrm{e}^{\cos 2x} \cos\big(\sin 2x\big)$?

Hint: Work smart, not hard.

Solution:

(i) Since $\frac{1}{|k|!}$ decays exponentially, the following expressions exist as absolutely convergent sum:

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}} \frac{1}{|k|!} \, \mathrm{e}^{+\mathrm{i}kx} = -1 + \sum_{k=0}^{\infty} \frac{1}{k!} \big(\mathrm{e}^{+\mathrm{i}x}\big)^k + \sum_{k=0}^{\infty} \frac{1}{k!} \big(\mathrm{e}^{-\mathrm{i}x}\big)^k \\ &= -1 + \mathrm{e}^{\mathrm{e}^{+\mathrm{i}x}} + \mathrm{e}^{\mathrm{e}^{-\mathrm{i}x}} = -1 + \mathrm{e}^{\cos x} \, \big(\mathrm{e}^{+\mathrm{i}\sin x} + \mathrm{e}^{-\mathrm{i}\sin x}\big) \\ &= -1 + 2\mathrm{e}^{\cos x} \, \cos\big(\sin x\big) \end{aligned}$$

(ii) Expressing g in terms of f,

$$\begin{aligned} g(x) &= \mathrm{e}^{\cos(x - \frac{\pi}{2})} \, \cos\big(\sin\big(x - \tfrac{\pi}{2}\big)\big) \\ &= \tfrac{1}{2} f\big(x - \tfrac{\pi}{2}\big) + \tfrac{1}{2}, \end{aligned}$$

yields the Fourier coefficients of g for $k \neq 0$,

$$\mathcal{F}\big(f\big(\cdot - \tfrac{\pi}{2}\big)\big)(k) = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}k} \, \hat{f}(k) = \frac{\mathrm{i}^k}{|k|!},$$

i. e. we have

$$\hat{g}(k) = \begin{cases} 1 & k = 0 \\ \frac{\mathrm{i}}{2(|k|!)} & k \in \mathbb{Z} \setminus \{0\} \end{cases}.$$

(iii) Since $h(x) = \frac{1}{2} f(2x) + \frac{1}{2}$, we can express the Fourier coefficients for $k \neq 0$ as

$$\begin{aligned} \hat{h}(k) &= \widehat{f(2\,\cdot\,)}(k) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}kx} \, f(2x) = \frac{1}{2\pi} \int_{-2\pi}^{+2\pi} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} \, f(x) \\ &= \frac{1}{4\pi} \int_{-2\pi}^{-\pi} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} \, f(x) + \frac{1}{4\pi} \int_{-\pi}^{+\pi} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} \, f(x) + \frac{1}{4\pi} \int_{+\pi}^{+2\pi} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} \, f(x) \\ &= \frac{1}{4\pi} \int_{-\pi}^{+\pi} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} \, f(x - \pi) + \frac{1}{4\pi} \int_{-\pi}^{+\pi} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} \, f(x) \\ &= \tfrac{1}{2} \widehat{f(\cdot - \pi)}(k) + \tfrac{1}{2} \hat{f}(k) = \tfrac{1}{2} \big(\mathrm{e}^{+\mathrm{i}\pi} + 1\big) \, \hat{f}(k) \\ &= \begin{cases} \hat{f}(k/2) & k \in 2\mathbb{Z} \\ 0 & k \in 2\mathbb{Z} + 1 \end{cases}. \end{aligned}$$

Hence, we obtain

$$\hat{h}(k) = \begin{cases} 1 & k = 0 \\ \frac{1}{2(|k|!)} & k \in 2\mathbb{Z} \\ 0 & k \in 2\mathbb{Z} + 1 \end{cases}.$$

32. Density of $\mathcal{C}^\infty(\mathbb{T}^n) \subset L^1(\mathbb{T}^n)$

Prove that $\mathcal{C}^\infty(\mathbb{T}^n)$ is dense in $L^1(\mathbb{T}^n)$.

Solution:

Since continuous functions on the compact $\mathbb{T}^n$ are bounded, $\mathcal{C}^\infty(\mathbb{T}^n) \subset L^\infty(\mathbb{T}^n)$, they are integrable by Lemma 6.1.6.

Moreover, trigonometric polynomials are smooth and dense in $L^1(\mathbb{T}^n)$,

$$\mathrm{Pol}(\mathbb{T}^n) \subset \mathcal{C}^\infty(\mathbb{T}^n) \subset L^1(\mathbb{T}^n),$$

and hence $\mathcal{C}^\infty(\mathbb{T}^n)$ is dense as a superset of a dense subset.

33. The wave equation on $[-\pi, +\pi]$ (22 points)

Solve the wave equation

$$\partial_t^2 u(t) - \partial_x^2 u(t) = 0$$

on $[-\pi, +\pi]$ by expanding $u(t)$ as a Fourier series.

(i) Give the form of the generic solution $u(t)$ if the initial conditions satisfy $u(0) = f \in L^2([-\pi, +\pi])$ and $\partial_t u(0) = g \in L^2([-\pi, +\pi])$.

(ii) Show that $u(t)$ from (i) is square integrable for all $t \in \mathbb{R}$.

(iii) What does $u(t, x)$ look like if the initial conditions $f, g \in L^2([-\pi, +\pi])$ from (i) satisfy Neumann boundary conditions? Does the time-evolved solution $u(t)$ satisfy Neumann boundary conditions?

(iv) Solve the initial value problem for $f(x) = |x|$ and $g(x) = 1$.

Solution:

(i) If we plug

$$u(t, x) \overset{[1]}{=} \sum_{k \in \mathbb{Z}} \hat{u}(t, k)\, \mathrm{e}^{+\mathrm{i}kx}$$

into the wave equation, we obtain an equation relating the Fourier coefficients,

$$\begin{aligned} 0 = \big(\partial_t^2 - \partial_x^2\big) u(t, x) &\overset{[1]}{=} \big(\partial_t^2 - \partial_x^2\big) \sum_{k \in \mathbb{Z}} \hat{u}(t, k)\, \mathrm{e}^{+\mathrm{i}kx} \\ &\overset{[1]}{=} \sum_{k \in \mathbb{Z}} \big(\partial_t^2 \hat{u}(t, k) + k^2 \hat{u}(t, k)\big)\, \mathrm{e}^{+\mathrm{i}kx} . \end{aligned}$$

Since $f, g \in L^2([-\pi, +\pi])$, their Fourier series which converge in the L^2-sense [1],

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}} \hat{f}(k)\, \mathrm{e}^{+\mathrm{i}kx}, \\ g(x) &= \sum_{k \in \mathbb{Z}} \hat{g}(k)\, \mathrm{e}^{+\mathrm{i}kx}. \end{aligned}$$

Hence, we arrive at a family of equations for the Fourier coefficients

$$\partial_t^2 \hat{u}(t, k) + k^2 \hat{u}(t, k) \overset{[1]}{=} 0$$

with initial conditions

$$\hat{u}(0, k) = \hat{f}(k), \qquad \partial_t \hat{u}(0, k) = \hat{g}(k). \qquad [1]$$

The solution is

$$\hat{u}(t, k) \overset{[1]}{=} \begin{cases} a_+(0) + a_-(0)\, t & k = 0 \\ a_+(k)\, \mathrm{e}^{+\mathrm{i}kt} + a_-(k)\, \mathrm{e}^{-\mathrm{i}kt} & k \in \mathbb{Z} \setminus \{0\} \end{cases}$$

where the coefficients $a_\pm(k)$ are determined from the initial conditions:

$$\hat{f}(k) = \hat{u}(0,k) \overset{[1]}{=} \begin{cases} a_+(0) & k = 0 \\ a_+(k) + a_-(k) & k \in \mathbb{Z} \setminus \{0\} \end{cases}$$

$$\hat{g}(k) = \partial_t \hat{u}(0,k) \overset{[1]}{=} \begin{cases} a_-(0) & k = 0 \\ \mathrm{i}k\,\big(a_+(k) - a_-(k)\big) & k \in \mathbb{Z} \setminus \{0\} \end{cases}$$

The coefficients $a_\pm(k)$ can be expressed in terms of $\hat{f}$ and $\hat{g}$:

$$a_+(0) = \hat{f}(0), \quad a_-(0) = \hat{g}(0) \qquad [1]$$

$$a_\pm(k) = \frac{\hat{f}(k)}{2} \mp \mathrm{i}\,\frac{\hat{g}(k)}{2k}, \qquad k \in \mathbb{Z} \setminus \{0\} \qquad [1]$$

Hence, the solution is

$$\begin{aligned} u(t,x) &\overset{[1]}{=} \hat{f}(0) + t\,\hat{g}(0) + \frac{1}{2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \Big(\big(\hat{f}(k) - \tfrac{\mathrm{i}}{k}\,\hat{g}(k)\big)\,\mathrm{e}^{+\mathrm{i}kt} + \big(\hat{f}(k) + \tfrac{\mathrm{i}}{k}\,\hat{g}(k)\big)\,\mathrm{e}^{-\mathrm{i}kt} \Big)\,\mathrm{e}^{+\mathrm{i}kx} \\ &= \hat{f}(0) + t\,\hat{g}(0) + \frac{1}{2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \Big(\big(\hat{f}(k) - \tfrac{\mathrm{i}}{k}\,\hat{g}(k)\big)\,\mathrm{e}^{+\mathrm{i}k(x+t)} + \big(\hat{f}(k) + \tfrac{\mathrm{i}}{k}\,\hat{g}(k)\big)\,\mathrm{e}^{+\mathrm{i}k(x-t)} \Big)\,. \end{aligned}$$

(ii) The square integrability of f and g implies that the Fourier coefficients $\hat{f}$ and $\hat{g}$ are square summable. Thus, the estimate

$$\begin{aligned} \Big| \tfrac{1}{2}\,\big(\hat{f}(k) - \tfrac{\mathrm{i}}{k}\,\hat{g}(k)\big)\,\mathrm{e}^{+\mathrm{i}kt} + \tfrac{1}{2}\,\big(\hat{f}(k) + \tfrac{\mathrm{i}}{k}\,\hat{g}(k)\big)\,\mathrm{e}^{-\mathrm{i}kt} \Big| &\overset{[1]}{\leq} \big|\hat{f}(k)\big| + \big|\tfrac{\mathrm{i}}{k}\,\hat{g}(k)\big| \\ &\overset{[1]}{\leq} \big|\hat{f}(k)\big| + \big|\hat{g}(k)\big| \end{aligned}$$

proves that $\hat{u}(t)$ is square summable. Hence, the Fourier series for $u(t)$ exists in the L^2-sense [1].

(iii) Looking at the arguments on pp. 45–46, we see that solutions can be written as a linear combination of

$$\cos\big(k\,\tfrac{x-\pi}{2}\big) = \begin{cases} (-1)^k\,\cos jx & k = 2j,\ j \in \mathbb{N}_0 \\ (-1)^k\,\sin\frac{(2j+1)x}{2} & k = 2j+1,\ j \in \mathbb{N}_0 \end{cases}$$

for $k \in \mathbb{N}_0$ [1]:

$$u(x) = \sum_{j=0}^{\infty} a_j\,\cos jx + \sum_{j=1}^{\infty} b_j\,\sin\tfrac{(2j+1)x}{2}.$$

The factor $\frac{1}{2}$ doubles the length of the interval from $[0,\pi]$ to $[0,2\pi]$ while $-\pi$ shifts the interval $[0,2\pi]$ to $[-\pi,+\pi]$. Using this expansion, it is easy to see that the boundary conditions are preserved (just repeat the arguments on pp. 45–46) [1].

However, in the Fourier basis this is less obvious: The Fourier transform of the $\sin\frac{(2j+1)x}{2} = \frac{\mathrm{i}}{2}\big(\mathrm{e}^{-\mathrm{i}\frac{(2j+1)}{2}x} - \mathrm{e}^{+\mathrm{i}\frac{(2j+1)}{2}x}\big)$ terms are

$$\Big(\mathcal{F}\sin\tfrac{(2j+1)x}{2}\Big)(k) = \frac{(-1)^{k+j+1}}{\pi}\,\frac{k}{k^2 - {(2j+1)^2}/{4}}. \qquad [1]$$

That means the kth Fourier coefficient is given by

$$\hat{u}(k) = \tfrac{1}{2} a_{|k|} + \sum_{j=1}^{\infty} b_j \frac{(-1)^{k+j+1}}{\pi} \frac{k}{k^2 - {}^{(2j+1)^2}/_4}. \qquad [1]$$

(iv) The function $f(x) = |x|$ agrees almost everywhere with the function h from problem 30 (iii), and hence, the Fourier coefficients agree [1]. Thus, the solution is

$$\begin{aligned} u(t,x) &\overset{[1]}{=} \frac{\pi}{2} + t - \frac{1}{2} \sum_{k \in 2\mathbb{Z}+1} \frac{2}{\pi k^2} \left(\mathrm{e}^{+\mathrm{i}kt} + \mathrm{e}^{-\mathrm{i}kt}\right) \mathrm{e}^{+\mathrm{i}kx} \\ &\overset{[1]}{=} \frac{\pi}{2} + t - \sum_{k \in \mathbb{Z}} \frac{2}{\pi (2k+1)^2} \big(\cos(2k+1)t\big)\, \mathrm{e}^{+\mathrm{i}(2k+1)x}. \end{aligned}$$