UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics (APM 421 H)

Winter 2014
Problem Sheet 9
(2014.11.07)

Homework Problems

26. Boundedness of the spectrum of an operator

Let $T \in \mathcal{B}(\mathcal{X})$ where $\mathcal{X}$ is a Banach space. Show that

$$\sigma(T) \subseteq \big\{ z \in \mathbb{C} \mid |z| \leq \|T\| \big\}.$$

Hint: Look at the *resolvent* set.

27. Boundedness of multiplication operators

Let $V(\hat{x})$ be the multiplication operator on $L^2(\mathbb{R}^d)$ associated to the function V. Show that $V(\hat{x})$ is bounded if and only if $V \in L^\infty(\mathbb{R}^d)$.

28. Selfadjointness of Schrödinger operators

Let $H = -\Delta_x + V$ be the Schrödinger operator on $L^2(\mathbb{R}^d)$ with potential $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ equipped with domain $\mathcal{D}(H) = H^2(\mathbb{R}^d)$. Show that H is selfadjoint.

29. A quantum two-body system

Consider two identical quantum particles of mass m which are interacting with one another via Coulomb repulsion. Then the Schrödinger operator which describes the two is an extension of

$$H_0 = \frac{1}{2m}\big(-\mathrm{i}\nabla_{x_1}\big)^2 + \frac{1}{2m}\big(-\mathrm{i}\nabla_{x_2}\big)^2 + \frac{e^2}{|x_1 - x_2|}$$

endowed with domain $\mathcal{D}(H_0) = \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^6) \subset L^2(\mathbb{R}^6)$. Here, $x_1, x_2 \in \mathbb{R}^3$ are the positions of particles 1 and 2, respectively. We will denote the selfadjoint extension of H_0 with H (establishing the existence of H will be done in part (ii) below).

(i) Write H_0 in terms of center-of-mass coordinate $x_c = \frac{1}{2}(x_1 + x_2)$ and relative coordinate $r = x_1 - x_2$.

(ii) Show that H_0 is essentially selfadjoint. What is the domain of the selfadjoint extension H?

(iii) Does the selfadjoint extension H have eigenvalues below 0? Justify your answer.

(iv) What do low-energy states look like?

Define the fermionic subspace as

$$\mathcal{H}_{\mathrm{f}} := \Big\{ \varphi \in L^2(\mathbb{R}^6) \;\Big|\; \varphi(x_2, x_1) = -\varphi(x_1, x_2) \text{ almost everywhere} \Big\},$$

and denote the restriction of H to $\mathcal{D}(H) \cap \mathcal{H}_{\mathrm{f}}$ with H_{f}.

(v) Show that H_{f} maps $\mathcal{D}(H) \cap \mathcal{H}_{\mathrm{f}}$ to $\mathcal{H}_{\mathrm{f}}$. How do you interpret this fact physically?

(vi) Explain in what sense H_{f} defines a selfadjoint operator.

(vii) Show that $\inf \sigma(H) \leq \inf \sigma(H_{\mathrm{f}})$. How do you interpret this fact physically?

Hand in home work on: Friday, 17 November 2014, before class