UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 9
(2013.11.14)

The discrete Fourier transform

Homework Problems

30. The Fourier transform of the sawtooth and the tent function

Consider the function $f(x) := \max\{0, x\}$ on $(-\pi, +\pi]$. Determine the Fourier coefficients, their asymptotic behavior for large $|k|$ of

(i) f,

(ii) g with $g(x) = f(-x)$ and

(iii) $h = f + g$.

In each of the cases, sketch the graph and give the first few terms of the sin and cos representation.

Hint: Work smart, not hard.

31. Fourier series of particular functions

(i) Which periodic function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the Fourier coefficients $\hat{f}(k) = \frac{1}{|k|!}$, $k \in \mathbb{Z}$?

(ii) What are the Fourier coefficients of $g(x) = \mathrm{e}^{\sin x} \cos\bigl(\cos x\bigr)$?

(iii) What are the Fourier coefficients of $h(x) = \mathrm{e}^{\cos 2x} \cos\bigl(\sin 2x\bigr)$?

Hint: Work smart, not hard.

32. Density of $\mathcal{C}^\infty(\mathbb{T}^n) \subset L^1(\mathbb{T}^n)$

Prove that $\mathcal{C}^\infty(\mathbb{T}^n)$ is dense in $L^1(\mathbb{T}^n)$.

33. The wave equation on $[-\pi, +\pi]$

Solve the wave equation

$$\partial_t^2 u(t) - \partial_x^2 u(t) = 0$$

on $[-\pi, +\pi]$ by expanding $u(t)$ as a Fourier series.

(i) Give the form of the generic solution $u(t)$ if the initial conditions satisfy $u(0) = f \in L^2([-\pi, +\pi])$ and $\partial_t u(0) = g \in L^2([-\pi, +\pi])$.

(ii) Show that $u(t)$ from (i) is square integrable for all $t \in \mathbb{R}$.

(iii) What does $u(t, x)$ look like if the initial conditions $f, g \in L^2([-\pi, +\pi])$ from (i) satisfy Neumann boundary conditions? Does the time-evolved solution $u(t)$ satisfy Neumann boundary conditions?

(iv) Solve the initial value problem for $f(x) = |x|$ and $g(x) = 1$.

Hand in home work on: Thursday, 21 November 2013, before class