UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Solutions 10
(2014.11.14)

Functional Calculus

Homework Problems

30. Functional calculus for matrices revisited (30 points)

Let $H = h_0 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^3 h_j \, \sigma_j$ be a hermitian 2×2 matrix written in terms of the Pauli matrices σ_1, σ_2 and σ_3.

(i) Compute the projection-valued measure associated to H.

(ii) Show that the functional calculus introduced on Sheet 01 coincides with the functional calculus from Chapter 6 of the lecture notes.

(iii) Prove that the following inequality is *false*:

$$\Big\lvert (\sigma_3 + \mathrm{id}_{\mathbb{C}^2}) + (\sigma_1 - \mathrm{id}_{\mathbb{C}^2}) \Big\rvert \leq \lvert \sigma_3 + \mathrm{id}_{\mathbb{C}^2} \rvert + \lvert \sigma_1 - \mathrm{id}_{\mathbb{C}^2} \rvert$$

(iv) Show that the analogous inequality for the traces *does* hold true:

$$\mathrm{Tr} \Big\lvert (\sigma_3 + \mathrm{id}_{\mathbb{C}^2}) + (\sigma_1 - \mathrm{id}_{\mathbb{C}^2}) \Big\rvert \leq \mathrm{Tr} \, \lvert \sigma_3 + \mathrm{id}_{\mathbb{C}^2} \rvert + \mathrm{Tr} \, \lvert \sigma_1 - \mathrm{id}_{\mathbb{C}^2} \rvert$$

Solution:

(i) The spectrum is purely discrete, $\sigma(H) = \sigma_{\mathrm{disc}}(H) = \{E_-, E_+\}$ [1], where the eigenvalues are $E_\pm = h_0 \pm \sqrt{h_1^2 + h_2^2 + h_3^2}$ [1]. That means the spectral measure is pure point since its support is the discrete set $\{E_-, E_+\}$. Thus, the eigenprojections

$$P_\pm \overset{[1]}{=} \frac{1}{2} \left(\mathrm{id}_{\mathbb{C}^2} + \frac{\sum_{j=1}^3 h_j \, \sigma_j}{\sqrt{h_1^2 + h_2^2 + h_3^2}} \right)$$

are related to the projection-valued measure via

$$P(\Lambda) \overset{[1]}{=} \big(P_- + P_+ \big) \, P(\Lambda) \overset{[1]}{=} 1_\Lambda(\{E_-\}) \, P_- + 1_\Lambda(\{E_+\}) \, P_+$$

as $P_- + P_+ = \mathrm{id}_{\mathbb{C}^2}$ [1].

(ii) In Problem 02 of Sheet 01, we have defined

$$f(H) \overset{[1]}{:=} f(E_+) \, P_+ + f(E_-) \, P_-$$

where $f : \mathbb{R} \longrightarrow \mathbb{C}$ was a suitable function.

Compared to the definition in Chapter 06,

$$f(H) \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}P(\lambda)\, f(\lambda),$$

we see with the help of (i) that the two definitions coincide:

$$\begin{aligned} f(H) &\overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}P(\lambda)\, f(\lambda) \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}\lambda\, \big(\delta(\lambda - E_-)\, P_- + \delta(\lambda - E_+)\, P_+\big)\, \lambda \\ &\overset{[1]}{=} f(E_+)\, P_+ + f(E_-)\, P_- \end{aligned}$$

(iii) We will repeatedly use the formulas from Sheet 01. Let us start with the left-hand side: the eigenvalues of the matrix on the left $\sigma_3 + \mathrm{id}_{\mathbb{C}^2} + \sigma_1 - \mathrm{id}_{\mathbb{C}^2} = \sigma_1 + \sigma_3$ are $E^{\text{left}}_{\pm} = \pm\sqrt{1^2 + 1^2} = \pm\sqrt{2}$ [1] with eigenprojections

$$P^{\text{left}}_{\pm} \overset{[1]}{=} \frac{1}{2}\left(\mathrm{id}_{\mathbb{C}^2} \pm \frac{\sigma_1 + \sigma_3}{\sqrt{2}}\right).$$

Thus, the absolute value of the left-hand side is

$$\Big|\big(\sigma_3 + \mathrm{id}_{\mathbb{C}^2}\big) + \big(\sigma_1 - \mathrm{id}_{\mathbb{C}^2}\big)\Big| \overset{[1]}{=} \big|{+\sqrt{2}}\big|\, P^{\text{left}}_{+} + \big|{-\sqrt{2}}\big|\, P^{\text{left}}_{-} \overset{[1]}{=} \sqrt{2}\, \mathrm{id}_{\mathbb{C}^2}.$$

Now to the right-hand side: eigenvalues and eigenprojections of the first term are $E^{\text{right},1}_{\pm} = 1 \pm 1$ [1] and

$$P^{\text{right},1}_{\pm} \overset{[1]}{=} \tfrac{1}{2}\big(\mathrm{id}_{\mathbb{C}^2} \pm \sigma_3\big)$$

while those of the second term are $E^{\text{right},2}_{\pm} = -1 \pm 1$ [1]

$$P^{\text{right},2}_{\pm} \overset{[1]}{=} \tfrac{1}{2}\big(\mathrm{id}_{\mathbb{C}^2} \mp \sigma_1\big).$$

That means their absolute values are

$$\begin{aligned} \big|\sigma_3 + \mathrm{id}_{\mathbb{C}^2}\big| &\overset{[1]}{=} |2| \cdot P^{\text{right},1}_{+} + |0| \cdot P^{\text{right},1}_{-} \overset{[1]}{=} \mathrm{id}_{\mathbb{C}^2} + \sigma_3, \\ \big|\sigma_1 - \mathrm{id}_{\mathbb{C}^2}\big| &\overset{[1]}{=} |0| \cdot P^{\text{right},2}_{+} + |-2| \cdot P^{\text{right},2}_{-} \overset{[1]}{=} \mathrm{id}_{\mathbb{C}^2} + \sigma_1. \end{aligned}$$

Consequently, the eigenvalues of their sum

$$\big|\sigma_3 + \mathrm{id}_{\mathbb{C}^2}\big| + \big|\sigma_1 - \mathrm{id}_{\mathbb{C}^2}\big| \overset{[1]}{=} 2\mathrm{id}_{\mathbb{C}^2} + \sigma_1 + \sigma_3$$

are $E^{\text{sum}}_{\pm} = 2 \pm \sqrt{2}$ [1], and since $E^{\text{sum}}_{-} = 2 - \sqrt{2} < \sqrt{2}$ [1], the inequality is necessarily false [1].

(iv) The trace of the matrix on the left is $2\sqrt{2}$ while that of the matrix on the right is 4, and hence, the inequality involving the traces is satisfied,

$$\mathrm{Tr}\, \Big|\big(\sigma_3 + \mathrm{id}_{\mathbb{C}^2}\big) + \big(\sigma_1 - \mathrm{id}_{\mathbb{C}^2}\big)\Big| \overset{[1]}{=} 2\sqrt{2} \overset{[1]}{\leq} 4 \overset{[1]}{=} \mathrm{Tr}\, \big|\sigma_3 + \mathrm{id}_{\mathbb{C}^2}\big| + \mathrm{Tr}\, \big|\sigma_1 - \mathrm{id}_{\mathbb{C}^2}\big|.$$

31. Projections and functional calculus (15 points)

Let P be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Show that P is an orthogonal projection if and only if $\sigma(P) \subseteq \{0, 1\}$.

Solution:

"⇒:" Suppose P is an orthogonal projection. Orthogonal projections are bounded, and the spectrum is closed and bounded by Problem 26. Let us define the bounded function

$$f(\lambda) \overset{[1]}{=} \begin{cases} \lambda^2 & |\lambda| \leq 2 \left\| P \right\| \\ 0 & \text{else} \end{cases} .$$

Then we can use functional calculus to express P^2 as $f(P)$,

$$\begin{aligned} P^2 &\overset{[1]}{=} f(P) \overset{[1]}{=} \int_{\mathbb{R}} 1_{\mathrm{d}\lambda}(P)\, f(\lambda) \overset{[1]}{=} \int_{\sigma(P)} 1_{\mathrm{d}\lambda}(P)\, \lambda^2 \\ &\overset{!}{=} P \overset{[1]}{=} \int_{\sigma(P)} 1_{\mathrm{d}\lambda}(P)\, \lambda, \end{aligned}$$

and deduce that on $\sigma(P)$ the equation $\lambda^2 = \lambda$ holds [1]. Evidently, the only two solutions to this equation are 0 and 1, i. e. $\sigma(P) \subseteq \{0, 1\}$ [1].

"⇐:" Suppose $\sigma(P) \subseteq \{0, 1\}$ where P is selfadjoint [1]. As the spectrum is bounded, so is the operator P [1]. Then we can use the same reasoning as above in reverse: let f be as above: since $\lambda^2 = \lambda$ on $\sigma(P)$ [1], we deduce $P^2 = P$ from

$$\begin{aligned} P^2 &\overset{[1]}{=} f(P) \overset{[1]}{=} \int_{\mathbb{R}} 1_{\mathrm{d}\lambda}(P)\, f(\lambda) \overset{[1]}{=} \int_{\sigma(P)} 1_{\mathrm{d}\lambda}(P) \underbrace{\lambda^2}_{=\lambda \text{ on } \sigma(P)} \\ &\overset{[1]}{=} \int_{\sigma(P)} 1_{\mathrm{d}\lambda}(P)\, \lambda \overset{[1]}{=} P. \end{aligned}$$

Hence, $P = P^* = P^2$ is an orthogonal projection [1].

32. The semirelativistic kinetic energy (32 points)

Consider a semirelativistic quantum particle subjected to a magnetic field $B = \nabla_x \times A$. (Semirelativistic here means that you are in an energy regime where one cannot yet create particle-antiparticle pairs but the energies are high enough so that one needs to take the relativistic kinetic energy.)

Assuming the vector potential $A \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is smooth, the hamiltonian

$$H^A = \sqrt{m^2 + \bigl(-\mathrm{i}\nabla_x - A(\hat{x})\bigr)^2}.$$

The purpose of this problem is to rigorously define H^A. You may use without proof that kinetic momentum $P^A_j := -\mathrm{i}\partial_{x_j} - A_j(\hat{x})$ and $(P^A)^2$ (endowed with the correct domains) define selfadjoint operators.

(i) Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ be a phase function and $\mathrm{e}^{+\mathrm{i}\phi}$ the associated unitary. Show that kinetic momentum is gauge-covariant,

$$P^{A+\nabla_x\phi} = \mathrm{e}^{+\mathrm{i}\phi}\, P^A\, \mathrm{e}^{-\mathrm{i}\phi}.$$

(ii) Find a less laborious way to prove $(P^{A+\nabla_x\phi})^2 = \mathrm{e}^{+\mathrm{i}\phi}\,(P^A)^2\,\mathrm{e}^{-\mathrm{i}\phi}$. Work smart, not hard.

(iii) Define the semirelativistic kinetic energy $\sqrt{m^2 + (P^A)^2}$.

(iv) Prove $\sqrt{m^2 + (P^A)^2} \geq \bigl|P^A\bigr|$.

(v) Show $\sqrt{m^2 + (P^{A+\nabla_x\phi})^2} = \mathrm{e}^{+\mathrm{i}\phi}\,\sqrt{m^2 + (P^A)^2}\,\mathrm{e}^{-\mathrm{i}\phi}$.

Solution:

(i) For $\psi \in \mathcal{D}\bigl(P^{A+\nabla_x\phi}_j\bigr) := \mathrm{e}^{+\mathrm{i}\phi}\mathcal{D}(P^A_j)$, we compute

$$\begin{aligned}
\mathrm{e}^{+\mathrm{i}\phi}\, P^A_j\, \mathrm{e}^{-\mathrm{i}\phi}\psi &\overset{[1]}{=} \mathrm{e}^{+\mathrm{i}\phi}\bigl(-\mathrm{i}\partial_{x_j} - A_j(\hat{x})\bigr)\mathrm{e}^{-\mathrm{i}\phi}\psi \\
&\overset{[1]}{=} \mathrm{e}^{+\mathrm{i}\phi}\bigl(\mathrm{e}^{-\mathrm{i}\phi}\,(-\mathrm{i}\partial_{x_j}\psi) - \mathrm{i}(\partial_{x_j}\mathrm{e}^{-\mathrm{i}\phi})\psi - A_j(\hat{x})\bigr) \\
&\overset{[1]}{=} \bigl(-\mathrm{i}\partial_{x_j} - A_j(\hat{x}) - \mathrm{i}\partial_{x_j}\phi(\hat{x})\bigr)\psi \\
&= P^{A+\nabla_x\phi}_j\psi.
\end{aligned}$$

(ii) Writing $(P^A)^2 = P^A \cdot P^A$ and using the result from (i), we obtain

$$\begin{aligned}
\mathrm{e}^{+\mathrm{i}\phi}\,(P^A)^2\,\mathrm{e}^{-\mathrm{i}\phi} &\overset{[1]}{=} \mathrm{e}^{+\mathrm{i}\phi}\,P^A\mathrm{e}^{-\mathrm{i}\phi} \cdot \mathrm{e}^{+\mathrm{i}\phi}\,P^A\mathrm{e}^{-\mathrm{i}\phi} \\
&\overset{[1]}{=} P^{A+\nabla_x\phi} \cdot P^{A+\nabla_x\phi} \overset{[1]}{=} \bigl(P^{A+\nabla_x\phi}\bigr)^2.
\end{aligned}$$

Note that the unitary $\mathrm{e}^{+\mathrm{i}\phi}$ relates the domains $\mathcal{D}\bigl((P^A)^2\bigr)$ and $\mathcal{D}\bigl((P^{A+\nabla_x\phi})^2\bigr) \overset{[1]}{=} \mathrm{e}^{+\mathrm{i}\phi}\mathcal{D}\bigl((P^A)^2\bigr)$.

(iii) The semirelativistic kinetic energy is defined via functional calculus: $(P^A)^2 \geq 0$ is non-negative [1] and selfadjoint [1] with domain $\mathcal{D}\bigl((P^A)^2\bigr)$, and consequently, the semirelativistic kinetic energy is the operator

$$\sqrt{m^2 + (P^A)^2} \overset{[2]}{:=} \int_0^{+\infty} 1_{\mathrm{d}\lambda}\bigl((P^A)^2\bigr)\,\sqrt{m^2+\lambda}$$

endowed with domain

$$\mathcal{D}\Bigl(\sqrt{m^2 + (P^A)^2}\Bigr) \overset{[2]}{:=} \Bigl\{\psi \in L^2(\mathbb{R}^3) \;\big|\; \int_0^{+\infty} \bigl\langle \psi, 1_{\mathrm{d}\lambda}\bigl((P^A)^2\bigr)\psi\bigr\rangle\,(m^2+\lambda) < \infty\Bigr\}.$$

(Since the spectrum $\sigma\bigl((P^A)^2\bigr) \subseteq [0, +\infty)$ is non-negative, we can omit the absolute value sign.)

(iv) We define

$$\left|P^A\right| \overset{[2]}{:=} \sqrt{(P^A)^2} \overset{[1]}{=} \int_0^{+\infty} 1_{\mathrm{d}\lambda}\big((P^A)^2\big)\ |\lambda|$$

analogously to the semirelativistic kinetic energy endowed with domain

$$\begin{aligned}\mathcal{D}\big(\left|P^A\right|\big) &\overset{[1]}{:=} \Big\{\psi \in L^2(\mathbb{R}^3) \ \Big|\ \int_0^{+\infty} \langle \psi, 1_{\mathrm{d}\lambda}\big((P^A)^2\big)\psi\rangle\, \lambda < \infty\Big\} \\ &\overset{[1]}{=} \Big\{\psi \in L^2(\mathbb{R}^3) \ \Big|\ \int_0^{+\infty} \langle \psi, 1_{\mathrm{d}\lambda}\big((P^A)^2\big)\psi\rangle\, (m^2+\lambda) < \infty\Big\} \\ &\overset{[1]}{=} \mathcal{D}\Big(\sqrt{m^2+(P^A)^2}\Big).\end{aligned}$$

The domain of $\left|P^A\right|$ coincides with the domain of $\sqrt{m^2+(P^A)^2}$ [1], and consequently, the gauge-covariance of $\sqrt{m^2+(P^A)^2} \geq \left|P^A\right|$ follows from $\sqrt{m^2+\lambda} \geq \sqrt{\lambda}$ on the level of functions [1] and functional calculus [1].

(v) The covariance of $(P^A)^2$ proven in (ii) also implies that the projection-valued measure of this operator is also gauge-covariant,

$$1_\Lambda\Big(\big(P^{A+\nabla_x\phi}\big)^2\Big) = \mathrm{e}^{+\mathrm{i}\phi}\, 1_\Lambda\big((P^A)^2\big)\, \mathrm{e}^{-\mathrm{i}\phi}. \tag{1}$$

And if we assume for a moment that we have shown the gauge-covariance of the projection-valued measure, then also the semirelativistic kinetic energy inherits the gauge-covariance,

$$\begin{aligned}\sqrt{m^2+\big(P^{A+\nabla_x\phi}\big)^2} &\overset{[1]}{=} \int_{\mathbb{R}} 1_{\mathrm{d}\lambda}\Big(\big(P^{A+\nabla_x\phi}\big)^2\Big)\, \sqrt{m^2+\lambda} \\ &\overset{[1]}{=} \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}\phi}\, 1_\Lambda\big((P^A)^2\big)\, \mathrm{e}^{-\mathrm{i}\phi}\, \sqrt{m^2+\lambda} \\ &\overset{[1]}{=} \mathrm{e}^{+\mathrm{i}\phi}\, \sqrt{m^2+(P^A)^2}\, \mathrm{e}^{-\mathrm{i}\phi}.\end{aligned}$$

Hence, it remains to show equation (1). From the gauge-covariance of the resolvent,

$$\Big(\big(P^{A+\nabla_x\phi}\big)^2 - z\Big)^{-1} \overset{[1]}{=} \mathrm{e}^{+\mathrm{i}\phi}\, \big((P^A)^2 - z\big)^{-1}\, \mathrm{e}^{-\mathrm{i}\phi},$$

and the Herglotz representation theorem,

$$\begin{aligned}\Big\langle \mathrm{e}^{-\mathrm{i}\phi}\psi, \big((P^A)^2 - z\big)^{-1} \mathrm{e}^{-\mathrm{i}\phi}\psi\Big\rangle &\overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}\mu^A_{\mathrm{e}^{-\mathrm{i}\phi}\psi}(\lambda)\, (\lambda - z)^{-1} \\ &\overset{[1]}{=} \Big\langle \psi, \Big(\big(P^{A+\nabla_x\phi}\big)^2 - z\Big)^{-1}\psi\Big\rangle \\ &\overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}\mu^{A+\nabla_x\phi}_{\psi}(\lambda)\, (\lambda - z)^{-1},\end{aligned}$$

we can relate the measures of $(P^A)^2$ and vector $\mathrm{e}^{-\mathrm{i}\phi}\psi$ to that of $\big(P^{A+\nabla_x\phi}\big)^2$ and vector ψ (as indicated by the superscripts on the measures) [1]. Thus, we deduce the gauge-covariance for the projection-valued measure,

$$\begin{aligned}\Big\langle \psi, \mathrm{e}^{-\mathrm{i}\phi}\, 1_\Lambda\big((P^A)^2\big)\, \mathrm{e}^{-\mathrm{i}\phi}\psi\Big\rangle &\overset{[1]}{=} \int_\Lambda \mathrm{d}\mu^A_{\mathrm{e}^{-\mathrm{i}\phi}\psi}(\lambda) \overset{[1]}{=} \int_\Lambda \mathrm{d}\mu^{A+\nabla_x\phi}_{\psi}(\lambda) \\ &\overset{[1]}{=} \Big\langle \psi, 1_\Lambda\Big(\big(P^{A+\nabla_x\phi}\big)^2\Big)\psi\Big\rangle.\end{aligned}$$

33. Resolution of the identity (30 points)

Suppose H be a selfadjoint operator on a Hilbert space $\mathcal{H}$ whose spectrum is purely discrete, i. e. $\sigma(H) = \sigma_{\text{disc}}(H) = \{E_n\}_{n\in\mathbb{N}}$. Here, the eigenvalues E_n are repeated according to their multiplicity.

(i) Express the projection-valued measure in terms of the eigenfunctions of φ_n.

(ii) Write out $H = \int_{\mathbb{R}} \mathrm{d}P(\lambda)\,\lambda$ explicitly.

(iii) Prove that there exists a resolution of the identity with respect to the eigenfunctions φ_n,

$$\mathrm{id}_{\mathcal{H}} = \sum_{n\in\mathbb{N}} |\varphi_n\rangle\langle\varphi_n|.$$

Put another way, show that $\overline{\mathrm{span}\{\varphi_n\}_{n\in\mathbb{N}}} = \mathcal{H}$.

(iv) Show that H is necessarily unbounded.

Hint: Review the definitions of discrete and essential spectrum.

Solution:

(i) Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal set composed of eigenfunctions (in case of degeneracies, choose an orthonormal basis for each of the eigenspace). Then the projection-valued measure is defined through

$$\big\langle \psi, 1_\Lambda(H)\psi \big\rangle \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}\mu_\psi(\lambda)\, 1_\Lambda(\lambda) \overset{[1]}{=} \int_\Lambda \mathrm{d}\mu_\psi(\lambda) \tag{2}$$

We will now show that for $\psi \in \overline{\mathrm{span}\{\varphi_n\}_{E_n\in\Lambda}} =: \mathcal{H}_\Lambda$, the right-hand side is $\|\psi\|^2$ [1] while for $\psi \in \mathcal{H}_\Lambda^\perp$ we get 0 [1]: if φ_n is an eigenfunction to E_n, we compute

$$\big\langle \varphi_n, (H-z)^{-1}\varphi_n \big\rangle \overset{[1]}{=} (E_n - z)^{-1}\,\|\varphi_n\|^2 \overset{[1]}{=} (E_n - z)^{-1}.$$

On the other hand, the representation theorem for Herglotz functions relates the left-hand side to the measure $\mu_{\varphi_n}(\lambda)$

$$\big\langle \varphi_n, (H-z)^{-1}\varphi_n \big\rangle \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}\mu_{\varphi_n}(\lambda)\,(\lambda - z)^{-1} \overset{!,[1]}{=} (E_n - z)^{-1},$$

meaning that $\mathrm{d}\mu_{\varphi_n}(\lambda) = \delta(\lambda - E_n)\,\|\varphi_n\|^2 = \delta(\lambda - E_n)$ [1]. Hence, if $\psi \in \mathcal{H}_\Lambda$ then the right-hand side of (2) is necessarily $\|\psi\|^2$ [1] while if it is in the orthogonal complement, we get 0 [1]. Thus, we have shown

$$P(\Lambda) \overset{[1]}{=} 1_\Lambda(H) \overset{[1]}{=} \sum_{\{n\in\mathbb{N} \mid E_n\in\Lambda\}} |\varphi_n\rangle\langle\varphi_n| \overset{[1]}{=} \sum_{n\in\mathbb{N}} 1_\Lambda(E_n)\,|\varphi_n\rangle\langle\varphi_n|.$$

(ii) The projection-valued measure is the infinite sum of point measures, and hence, we obtain

$$H \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}P(\lambda)\,\lambda \overset{[1]}{=} \sum_{n\in\mathbb{N}} E_n\,|\varphi_n\rangle\langle\varphi_n|.$$

(iii) Lemma 6.1.10 tells us that

$$P(\Lambda) \overset{[1]}{=} 1_\Lambda(H) \overset{[1]}{=} \sum_{n\in\mathbb{N}} 1_\Lambda(E_n)\, |\varphi_n\rangle\langle\varphi_n|$$

is a projection-valued measure, and all projection valued measures satisfy

$$\mathrm{id}_{\mathcal{H}} \overset{[1]}{=} P(\mathbb{R}) \overset{[1]}{=} \sum_{n\in\mathbb{N}} 1_\mathbb{R}(E_n)\, |\varphi_n\rangle\langle\varphi_n| \overset{[1]}{=} \sum_{n\in\mathbb{N}} |\varphi_n\rangle\langle\varphi_n|.$$

This is equivalent to saying that $\{\varphi_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis of $\mathcal{H}$.

(iv) Assume H were bounded [1]. Then by Problem 26, we have $\sigma(H) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|H\|\}$ [1] and the spectrum is bounded [1]. Hence, the sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded [1] and consequently must have an accumulation point [1]. Such an accumulation point is part of the essential spectrum by Theorem 5.2.8 [1], and thus, $\sigma_{\text{ess}}(H) \neq \emptyset$ [1] which contradicts our assumption that the spectrum is purely discrete [1]. Hence, H has to be unbounded [1].