UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 10
(2013.11.21)

The discrete Fourier transform & Applications to 2×2 matrix problems

Homework Problems

34. The Fourier transform of various functions (8 points)

Compute the Fourier coefficients of the following functions on $[-\pi, +\pi]$ and characterize their asymptotic behavior for large $|k|$:

(i) $f(x) = 1 + x$

(ii) $g(x) = \sin 2x$

(iii) $h(x) = \begin{cases} +1 & x \in [0, +\pi] \\ 0 & x \in [-\pi, 0) \end{cases}$

(iv) $j(x) = \begin{cases} +1 & x \in [0, +\pi] \\ -1 & x \in [-\pi, 0) \end{cases}$

Solution:

(i) From $(\mathcal{F}1)(k) = \delta_{k,0}$ and

$$(\mathcal{F}x)(k) = \begin{cases} 0 & k = 0 \\ (-1)^k \, \frac{\mathrm{i}}{k} & k \in \mathbb{Z} \setminus \{0\} \end{cases},$$

we can immediately give the Fourier series of f as

$$(\mathcal{F}f)(k) \overset{[2]}{=} \begin{cases} 1 & k = 0 \\ (-1)^k \, \frac{\mathrm{i}}{k} & k \in \mathbb{Z} \setminus \{0\} \end{cases}.$$

The Fourier series decays as $1/|k|$.

(ii) Writing $\sin 2x = \frac{1}{\mathrm{i}2}\big(\mathrm{e}^{+\mathrm{i}2x} - \mathrm{e}^{-\mathrm{i}2x}\big)$ in terms of exponential functions immediately yields

$$(\mathcal{F}g)(k) \overset{[2]}{=} \begin{cases} -\frac{\mathrm{i}}{2} & k = 2 \\ +\frac{\mathrm{i}}{2} & k = -2 \\ 0 & \text{else} \end{cases}.$$

The Fourier series has only finitely many non-zero terms, i. e. it decays superpolynomially and superexponentially.

(iii) For $k = 0$, we obtain

$$(\mathcal{F}h)(0) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathrm{d}x\, h(x) = \frac{1}{2\pi} \int_{0}^{+\pi} \mathrm{d}x \overset{[1]}{=} \tfrac{1}{2},$$

while for $k \neq 0$, we get

$$\begin{aligned}(\mathcal{F}h)(k) &= \frac{1}{2\pi} \int_{0}^{+\pi} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} = \left[\frac{1}{2\pi} \frac{1}{-\mathrm{i}k} \mathrm{e}^{-\mathrm{i}kx}\right]_0^{+\pi} \\ &\overset{[1]}{=} \frac{\mathrm{i}\big((-1)^k - 1\big)}{2\pi k}.\end{aligned}$$

The Fourier series decays as $1/|k|$.

(iv) Noticing that $j(x) = 2h(x) - 1$, we deduce

$$(\mathcal{F}j)(k) \overset{[2]}{=} \begin{cases} 0 & k = 0 \\ \frac{\mathrm{i}((-1)^k - 1)}{\pi k} & k \in \mathbb{Z} \setminus \{0\} \end{cases}.$$

The Fourier series decays as $1/|k|$.

35. The Pauli matrices

Consider the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Prove $\sigma_j \, \sigma_k = \delta_{jk} \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \, \sigma_l$ where ϵ_{jkl} is the epsilon tensor.

(ii) Prove that any 2×2 matrix can be written as the linear combination of the identity and the three Pauli matrices with coefficients h_0 and $h = (h_1, h_2, h_3)$,

$$\mathrm{Mat}_{\mathbb{C}}(2) \ni A = (a_{jk})_{1 \leq j,k \leq 2} = h_0 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^{3} h_j \, \sigma_j =: \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma. \tag{1}$$

Hint: Use that $\mathrm{Mat}_{\mathbb{C}}(2)$ is finite-dimensional.

(iii) Now assume that the coefficients $h_0, \ldots, h_3$ in equation (1) are real. Show that then the resulting matrix $H = h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma$ is hermitian. Compute the eigenvalues $E_{\pm}(h_0, h)$ of H in terms of the coefficients h_0 and h.

(iv) Use (i) to prove that for real $h_0, \ldots, h_3$

$$P_{\pm}(h_0, h) = \frac{1}{2} \left(\mathrm{id}_{\mathbb{C}^2} \pm \frac{h \cdot \sigma}{|h|} \right), \qquad h \neq 0 \in \mathbb{R}^3, \ |h| := \sqrt{h_1^2 + h_2^2 + h_3^2},$$

are the projections onto the eigenspaces for the two eigenvalues $E_{\pm}(h_0, h)$ of H.

(v) Compute the trace of H.

Note: In physics especially, one frequently writes $h \cdot \sigma$ for $\sum_{j=1}^{3} h_j \, \sigma_j$ where $h = (h_1, h_2, h_3)$.

Solution:

(i) This follows from direct computation: for $j = k$ we obtain

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathrm{id}_{\mathbb{C}^2}$$

while for $j < k$

$$\begin{aligned}
\sigma_1 \, \sigma_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} +\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i} \, \sigma_3 \\
\sigma_1 \, \sigma_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i} \, \sigma_2 \\
\sigma_2 \, \sigma_3 &= \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & +\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} = \mathrm{i} \, \sigma_1
\end{aligned}$$

In other words, we have shown (i) for $j < k$.

To show (i) in the remaining cases, we use that the $\sigma_j = \sigma_j^*$ are hermitian matrices, and hence for $j < k$ we obtain

$$\begin{aligned}
\sigma_k \, \sigma_j = \bigl(\sigma_j \, \sigma_k \bigr)^* &= \left(\delta_{jk} \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \, \sigma_l \right)^* \\
&= \delta_{jk} \, \mathrm{id}_{\mathbb{C}^2} - \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \, \sigma_l = \delta_{jk} \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \epsilon_{kjl} \, \sigma_l .
\end{aligned}$$

This proves (i).

(ii) The vector space of 2×2 matrices is four-dimensional, $\dim \mathrm{Mat}_{\mathbb{C}}(2) = 4$, and seeing as the 4 vectors $\{\mathrm{id}_{\mathbb{C}^2}, \sigma_1, \sigma_2, \sigma_3\}$ are linearly independent, they form a basis of $\mathrm{Mat}_{\mathbb{C}}(2)$.

(iii) In case $h_0, \ldots, h_3$ are real,

$$\begin{aligned} H^* &= \big(h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma\big)^* = \overline{h_0} \, \mathrm{id}_{\mathbb{C}^2} + \overline{h} \cdot \sigma \\ &= h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma = H \end{aligned}$$

is hermitian and we can compute both eigenvalues: the characteristic polynomial of H is

$$\begin{aligned} \chi(\lambda) &= \det\big(\lambda \, \mathrm{id}_{\mathbb{C}^2} - H\big) = \det \begin{pmatrix} \lambda - h_0 - h_3 & h_1 - \mathrm{i}\, h_2 \\ h_1 + \mathrm{i}\, h_2 & \lambda - h_0 + h_3 \end{pmatrix} \\ &= \big((\lambda - h_0) - h_3\big) \big((\lambda - h_0) + h_3\big) - \big(h_1 - \mathrm{i} h_2\big)\big(h_1 + \mathrm{i}\, h_2\big) \\ &= (\lambda - h_0)^2 - \big(h_1^2 + h_2^2 + h_3^2\big) = (\lambda - h_0)^2 - |h|^2, \end{aligned}$$

and hence, the eigenvalues are $E_\pm(h_0, h) = h_0 \pm |h|$.

(iv) The product

$$H \, P_\pm = \big(h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma\big) \, P_\pm = h_0 \, P_\pm + \frac{1}{2} \left(h \cdot \sigma \pm \frac{(h \cdot \sigma)^2}{|h|} \right)$$

involves the square of $h \cdot \sigma$ which can be computed with the help of (i):

$$\begin{aligned} \big(h \cdot \sigma\big)^2 &= \sum_{j,k=1}^{3} h_j \, h_k \, \sigma_j \, \sigma_k \\ &= \sum_{j=1}^{3} h_j^2 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{\substack{j,k,l=1,2,3 \\ j \neq k}} h_j \, h_k \, \mathrm{i} \, \epsilon_{jkl} \, \sigma_l \\ &= |h|^2 \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \Bigg(\sum_{\substack{j,k=1,2,3 \\ j \neq k}} h_j \, h_k \, \mathrm{i} \, \epsilon_{jkl} \Bigg) \, \sigma_l = |h|^2 \, \mathrm{id}_{\mathbb{C}^2} \end{aligned}$$

Hence, we can factor out $E_\pm$ and obtain (iv):

$$\begin{aligned} H \, P_\pm &= h_0 \, P_\pm + \frac{1}{2} \left(h \cdot \sigma \pm \frac{|h|^2 \, \mathrm{id}_{\mathbb{C}^2}}{|h|} \right) \\ &= h_0 \, P_\pm \pm |h| \, \frac{1}{2} \left(\mathrm{id}_{\mathbb{C}^2} \pm \frac{h \cdot \sigma}{|h|} \right) \\ &= \big(h_0 \pm |h|\big) \, P_\pm = E_\pm \, P_\pm \end{aligned}$$

(v) The trace is just the sum over the diagonal elements of the matrices, and clearly, the Pauli matrices are all traceless. Hence, we compute

$$\begin{aligned} \mathrm{tr} H &= \mathrm{tr}\big(h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma\big) \\ &= h_0 \, \mathrm{tr} \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^{3} h_j \, \mathrm{tr} \, \sigma_j = 2 h_0. \end{aligned}$$

36. Functional calculus for 2×2 matrices

Let f be a piecewise continuous function and $H = H^*$ a hermitian 2×2 matrix. Then define

$$f(H) := \sum_{j=\pm} f(E_\pm)\, P_\pm \tag{2}$$

where $E_\pm$ are the eigenvalues of H and $P_\pm$ the two projections from problem 35.

(i) Compute $f(H)$ defined as in equation (2) for $H = h \cdot \sigma$, $h \neq 0$, and

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

(ii) Show that $f(H)$ for $f(x) = \mathrm{e}^{-\mathrm{i}tx}$ (defined via (2)) coincides with the matrix exponential, i. e.

$$f(H) = \mathrm{e}^{-\mathrm{i}th_0} \left(\cos\bigl(|h|\, t\bigr) - \frac{\mathrm{i}}{|h|} \sin\bigl(|h|\, t\bigr)\, h \cdot \sigma \right) = \mathrm{e}^{-\mathrm{i}tH} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!}\, H^n. \tag{3}$$

Hint: Use $\mathrm{e}^{-\mathrm{i}t(h_0 + h\cdot\sigma)} = \mathrm{e}^{-\mathrm{i}th_0}\, \mathrm{e}^{-\mathrm{i}th\cdot\sigma}$.

(iii) Assuming h_0, h_1, h_2, h_3 are real, compute $\psi(t)$ for the initial condition $\psi(0) = \psi_0 \in \mathbb{C}^2$:

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \bigl(h_2\, \sigma_2 + h_3\, \sigma_3\bigr)\psi(t)$

(b) $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = h_2\, \sigma_2 \psi(t)$

(c) $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \bigl(h_0\, \mathrm{id}_{\mathbb{C}^2} + h_3\, \sigma_3\bigr)\psi(t)$

Solution:

(i) $f(H) = f(|h|)\, P_+ + f(-|h|)\, P_- = P_+$

(ii) For $h = 0$, H is a scalar multiple of the identity matrix and equation (3) holds. So let us assume $h \neq 0$. Then we first compute the left-hand side:

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}tx}(H) &= \mathrm{e}^{-\mathrm{i}t(h_0+|h|)}\, P_+ + \mathrm{e}^{-\mathrm{i}t(h_0-|h|)}\, P_- \\ &= \frac{1}{2}\Bigl(\mathrm{e}^{-\mathrm{i}t(h_0+|h|)} + \mathrm{e}^{-\mathrm{i}t(h_0-|h|)}\Bigr)\, \mathrm{id}_{\mathbb{C}^2} + \frac{1}{2}\Bigl(\mathrm{e}^{-\mathrm{i}t(h_0+|h|)} - \mathrm{e}^{-\mathrm{i}t(h_0-|h|)}\Bigr)\, \frac{h\cdot\sigma}{|h|} \\ &= \mathrm{e}^{-\mathrm{i}th_0} \left(\cos\bigl(|h|\, t\bigr) - \frac{\mathrm{i}}{|h|} \sin\bigl(|h|\, t\bigr)\, h \cdot \sigma \right) \end{aligned}$$

To obtain the right-hand side, we note

$$\begin{aligned} \bigl(h\cdot\sigma\bigr)^2 &= \sum_{j,k=1,2,3} h_j\, h_k\, \sigma_j\, \sigma_k = \sum_{j=1,2,3} h_j^2\, \sigma_j^2 + \sum_{\substack{j,k=1,2,3 \\ j\neq k}} h_j\, h_k\, \sigma_j\, \sigma_k \\ &= \sum_{j=1,2,3} h_j^2\, \mathrm{id}_{\mathbb{C}^2} + \sum_{l=1,2,3} \sum_{\substack{j,k=1,2,3 \\ j\neq k}} h_j\, h_k\, \epsilon_{jkl}\, \sigma_l = h^2, \end{aligned}$$

and thus we identify a pattern in $\bigl(h\cdot\sigma\bigr)^n$:

$$\begin{aligned} \bigl(h\cdot\sigma\bigr)^{2n} &= |h|^{2n}\, \mathrm{id}_{\mathbb{C}^2} \\ \bigl(h\cdot\sigma\bigr)^{2n+1} &= \bigl(h\cdot\sigma\bigr)^{2n}\, h\cdot\sigma = |h|^{2n}\, h\cdot\sigma \end{aligned}$$

This means that we can compute the matrix exponential after splitting the sum into even and odd terms:

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}tH} &= \mathrm{e}^{-\mathrm{i}th_0}\,\mathrm{e}^{-\mathrm{i}th\cdot\sigma} = \mathrm{e}^{-\mathrm{i}th_0}\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}\big(h\cdot\sigma\big)^n \\
&= \mathrm{e}^{-\mathrm{i}th_0}\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^{2n}}{(2n)!}\big(h\cdot\sigma\big)^{2n} + \mathrm{e}^{-\mathrm{i}th_0}\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^{2n+1}}{(2n+1)!}\big(h\cdot\sigma\big)^{2n+1} \\
&= \mathrm{e}^{-\mathrm{i}th_0}\sum_{n=0}^{\infty}(-1)^n\frac{(|h|\,t)^{2n}}{(2n)!}\,\mathrm{id}_{\mathbb{C}^2} - \frac{\mathrm{i}}{|h|}\,\mathrm{e}^{-\mathrm{i}th_0}\sum_{n=0}^{\infty}(-1)^n\frac{(|h|\,t)^{2n+1}}{(2n+1)!}\,h\cdot\sigma \\
&= \mathrm{e}^{-\mathrm{i}th_0}\left(\cos\big(|h|\,t\big)\,\mathrm{id}_{\mathbb{C}^2} - \frac{\mathrm{i}}{|h|}\,\sin\big(|h|\,t\big)\,h\cdot\sigma\right)
\end{aligned}$$

Thus, left- and right-hand side agree.

(iii) (a)

$$\begin{aligned}
\psi(t) &= \mathrm{e}^{tH}\psi_0 = \mathrm{e}^{t|h|}\,P_+(0,0,h_2,h_3) + \mathrm{e}^{-t|h|}\,P_-(0,0,h_2,h_3) \\
&= \frac{1}{2}\big(\mathrm{e}^{+t|h|} + \mathrm{e}^{-t|h|}\big)\,\psi_0 + \frac{1}{2}\frac{\mathrm{e}^{+t|h|} - \mathrm{e}^{-t|h|}}{|h|}\,\big(h_2\,\sigma_2 + h_3\,\sigma_3\big)\psi_0 \\
&= \cosh\big(t\,|h|\big)\,\psi_0 + \sinh\big(t\,|h|\big)\,\big(h_2\,\sigma_2 + h_3\,\sigma_3\big)\psi_0
\end{aligned}$$

(b) $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0 = \cos\big(t|h_2|\big)\,\psi_0 - \mathrm{i}\,\frac{h_2}{|h_2|}\,\sin\big(t|h_2|\big)\,\sigma_2\psi_0$

(c) $\psi(t) = \mathrm{e}^{+\mathrm{i}tH}\psi_0 = \mathrm{e}^{+\mathrm{i}th_0}\,\cos\big(t|h_2|\big)\,\psi_0 + \mathrm{i}\,\mathrm{e}^{+\mathrm{i}th_0}\,\frac{h_3}{|h_3|}\,\sin\big(t|h_3|\big)\,\sigma_3\psi_0$

37. A simple model for graphene (12 points)

Consider the nearest-neighbor model for graphene

$$H = \begin{pmatrix} q_3 \, \mathrm{id}_{\ell^2(\mathbb{Z}^2)} & 1_{\ell^2(\mathbb{Z}^2)} + q_1 \, \mathfrak{s}_1 + q_2 \, \mathfrak{s}_2 \\ 1_{\ell^2(\mathbb{Z}^2)} + q_1 \, \mathfrak{s}_1^* + q_2 \, \mathfrak{s}_2^* & -q_3 \, \mathrm{id}_{\ell^2(\mathbb{Z}^2)} \end{pmatrix} .$$

Here, $q_1, q_2 \in \mathbb{R}$ are hopping amplitudes while $q_3 \in \mathbb{R}$ is the so-called stagger parameter. Repeat the analysis in Chapter 6.1.5.2:

(i) Compute the momentum representation $H^{\mathcal{F}} := \mathcal{F}^{-1} \, H \, \mathcal{F}$. What Hilbert space does this operator act on?

(ii) Find a matrix-valued function $T(k)$ so that $H^{\mathcal{F}} = T(\hat{k})$ is the multiplication operator associated to T.

(iii) Find the eigenvalues $E_{\pm}(k)$ and eigenprojections $P_{\pm}(k)$ of $T(k)$.

(iv) Compute the unitary evolution group $U^{\mathcal{F}}(t)$ for $H^{\mathcal{F}}$.

(v) Compute the unitary evolution group $U(t)$ in position representation.

(vi) **Voluntary:** Identify the parameter region where the eigenvalues are not separated by a gap, i. e. $\inf_{k \in \mathbb{T}^n} \bigl| E_+(k) - E_-(k) \bigr| = 0$.

Remark: This nearest-neighbor model with stagger was used to investigate the piezoelectric effect in graphene: *Topological Polarization in Graphene-like Systems*, G. De Nittis and M. Lein, J. Phys. A **46** no. 38, p. 385001, 2013

Solution:

(i) Using that $\mathcal{F}^{-1} \, \mathfrak{s}_j \, \mathcal{F} = \mathrm{e}^{+\mathrm{i}\hat{k}_j}$ [1], we obtain

$$\begin{aligned} H^{\mathcal{F}} &= \mathcal{F}^{-1} \, H \, \mathcal{F} \\ &\overset{[2]}{=} \begin{pmatrix} q_3 \, \mathrm{id}_{L^2(\mathbb{T}^2)} & \mathrm{id}_{L^2(\mathbb{T}^2)} + q_1 \, \mathrm{e}^{+\mathrm{i}\hat{k}_1} + q_2 \, \mathrm{e}^{+\mathrm{i}\hat{k}_2} \\ \mathrm{id}_{L^2(\mathbb{T}^2)} + q_1 \, \mathrm{e}^{-\mathrm{i}\hat{k}_1} + q_2 \, \mathrm{e}^{-\mathrm{i}\hat{k}_2} & -q_3 \, \mathrm{id}_{L^2(\mathbb{T}^2)} \end{pmatrix} \\ &= \begin{pmatrix} q_3 \, \mathrm{id}_{L^2(\mathbb{T}^2)} & \omega(\hat{k}) \\ \overline{\omega(\hat{k})} & -q_3 \, \mathrm{id}_{L^2(\mathbb{T}^2)} \end{pmatrix} \end{aligned}$$

where $\omega(k) = 1 + q_1 \, \mathrm{e}^{+\mathrm{i}k_1} + q_2 \, \mathrm{e}^{+\mathrm{i}k_2}$ is defined just as in the lecture. $H^{\mathcal{F}}$ is a bounded operator acting on $L^2(\mathbb{T}^2, \mathbb{C}^2)$ [1].

(ii) The matrix-valued function can be read off as

$$T(k) \overset{[1]}{=} \begin{pmatrix} q_3 & \omega(k) \\ \overline{\omega(k)} & -q_3 \end{pmatrix} \overset{[1]}{=} \mathrm{Re}\,\bigl(\omega(k)\bigr) \, \sigma_1 - \mathrm{Im}\,\bigl(\omega(k)\bigr) \, \sigma_2 + q_3 \, \sigma_3 .$$

(iii) The eigenvalues and eigenprojections have already been calculated in problem 35:

$$\begin{aligned} E_{\pm}(k) &\overset{[1]}{=} \pm \sqrt{q_3^2 + \lvert \omega(k) \rvert^2} \\ P_{\pm}(k) &\overset{[1]}{=} \frac{1}{2} \left(\mathrm{id}_{L^2(\mathbb{T}^2)} \pm \frac{\mathrm{Re}\,\bigl(\omega(k)\bigr) \, \sigma_1 - \mathrm{Im}\,\bigl(\omega(k)\bigr) \, \sigma_2 + q_3 \, \sigma_3}{E_+(k)} \right) \end{aligned}$$

(iv) Problem 36 (ii) has introduced an efficient way to compute the unitary time evolution in the momentum representation [1]:

$$U^{\mathcal{F}}(t) = \mathrm{e}^{-\mathrm{i}tH^{\mathcal{F}}} \overset{[1]}{=} \cos\big(E_+(\hat{k})t\big)\,\mathrm{id}_{L^2(\mathbb{T}^2,\mathbb{C}^2)} - \frac{\mathrm{i}\,\sin\big(E_+(\hat{k})t\big)}{E_+(\hat{k})}\,T(\hat{k})$$

(v) According to Proposition 6.1.5 (ii), $\mathcal{F}\big(f\,g\big) = \mathcal{F}f * \mathcal{F}g$ holds [1], and hence

$$\begin{aligned} U(t)\psi &\overset{[1]}{=} \mathcal{F}\,U^{\mathcal{F}}(t)\,\mathcal{F}^{-1}\psi \\ &\overset{[1]}{=} \mathcal{F}\left(\cos\big(E_+t\big)\,\mathrm{id}_{L^2(\mathbb{T}^2,\mathbb{C}^2)} - \frac{\mathrm{i}\,\sin\big(E_+t\big)}{E_+}\,T\right) * \psi \end{aligned}$$

(vi) The system has no gap if and only if $E_+(k)^2 = q_3^2 + |\omega(k)|^2 = 0$ for some $k \in \mathbb{T}^2$. This is zero if and only if $q_3 = 0$ and

$$\omega(k) = 1 + q_1\,\mathrm{e}^{-\mathrm{i}k_1} + q_2\,\mathrm{e}^{-\mathrm{i}k_2} \overset{!}{=} 0$$

The result is best explained in a graph (Figure 2 (b) in the aforementioned publication):