UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics
(APM 421 H)

Winter 2014
Problem Sheet 10
(2014.11.14)

Functional Calculus

Homework Problems

30. Functional calculus for matrices revisited

Let $H = h_0 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^3 h_j \, \sigma_j$ be a hermitian 2×2 matrix written in terms of the Pauli matrices σ_1, σ_2 and σ_3.

(i) Compute the projection-valued measure associated to H.

(ii) Show that the functional calculus introduced on Sheet 01 coincides with the functional calculus from Chapter 6 of the lecture notes.

(iii) Prove that the following inequality is *false*:

$$\Bigl| (\sigma_3 + \mathrm{id}_{\mathbb{C}^2}) + (\sigma_1 - \mathrm{id}_{\mathbb{C}^2}) \Bigr| \leq \bigl| \sigma_3 + \mathrm{id}_{\mathbb{C}^2} \bigr| + \bigl| \sigma_1 - \mathrm{id}_{\mathbb{C}^2} \bigr|$$

(iv) Show that the analogous inequality for the traces *does* hold true:

$$\mathrm{Tr} \Bigl| (\sigma_3 + \mathrm{id}_{\mathbb{C}^2}) + (\sigma_1 - \mathrm{id}_{\mathbb{C}^2}) \Bigr| \leq \mathrm{Tr} \bigl| \sigma_3 + \mathrm{id}_{\mathbb{C}^2} \bigr| + \mathrm{Tr} \bigl| \sigma_1 - \mathrm{id}_{\mathbb{C}^2} \bigr|$$

31. Projections and functional calculus

Let P be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Show that P is an orthogonal projection if and only if $\sigma(P) \subseteq \{0, 1\}$.

32. The semirelativistic kinetic energy

Consider a semirelativistic quantum particle subjected to a magnetic field $B = \nabla_x \times A$. (Semirelativistic here means that you are in an energy regime where one cannot yet create particle-antiparticle pairs but the energies are high enough so that one needs to take the relativistic kinetic energy.) Assuming the vector potential $A \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is smooth, the hamiltonian

$$H^A = \sqrt{m^2 + \bigl(-\mathrm{i} \nabla_x - A(\hat{x}) \bigr)^2}.$$

The purpose of this problem is to rigorously define H^A. You may use without proof that kinetic momentum $P_j^A := -\mathrm{i} \partial_{x_j} - A_j(\hat{x})$ and $(P^A)^2$ (endowed with the correct domains) define selfadjoint operators.

(i) Let $\phi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ be a phase function and $\mathrm{e}^{+\mathrm{i}\phi}$ the associated unitary. Show that kinetic momentum is gauge-covariant,

$$P^{A + \nabla_x \phi} = \mathrm{e}^{+\mathrm{i}\phi} \, P^A \, \mathrm{e}^{-\mathrm{i}\phi}.$$

(ii) Find a less laborious way to prove $(P^{A+\nabla_x \phi})^2 = \mathrm{e}^{+\mathrm{i}\phi} \, (P^A)^2 \, \mathrm{e}^{-\mathrm{i}\phi}$. Work smart, not hard.

(iii) Define the semirelativistic kinetic energy $\sqrt{m^2 + (P^A)^2}$.

(iv) Prove $\sqrt{m^2 + (P^A)^2} \geq \bigl| P^A \bigr|$.

(v) Show $\sqrt{m^2 + (P^{A+\nabla_x \phi})^2} = \mathrm{e}^{+\mathrm{i}\phi} \, \sqrt{m^2 + (P^A)^2} \, \mathrm{e}^{-\mathrm{i}\phi}$.

33. Resolution of the identity

Suppose H be a selfadjoint operator on a Hilbert space $\mathcal{H}$ whose spectrum is purely discrete, i. e. $\sigma(H) = \sigma_{\mathrm{disc}}(H) = \{E_n\}_{n \in \mathbb{N}}$. Here, the eigenvalues E_n are repeated according to their multiplicity.

(i) Express the projection-valued measure in terms of the eigenfunctions of φ_n.

(ii) Write out $H = \int_{\mathbb{R}} \mathrm{d}P(\lambda) \, \lambda$ explicitly.

(iii) Prove that there exists a resolution of the identity with respect to the eigenfunctions φ_n,

$$\mathrm{id}_{\mathcal{H}} = \sum_{n \in \mathbb{N}} |\varphi_n\rangle\langle\varphi_n|.$$

Put another way, show that $\mathrm{span}\{\varphi_n\}_{n \in \mathbb{N}} = \mathcal{H}$.

(iv) Show that H is necessarily unbounded.

Hint: Review the definitions of discrete and essential spectrum.

Hand in home work on: Friday, 21 November 2014, before class