UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 10
(2013.11.21)

The discrete Fourier transform & Applications to 2×2 matrix problems

Homework Problems

34. The Fourier transform of various functions

Compute the Fourier coefficients of the following functions on $[-\pi, +\pi]$ and characterize their asymptotic behavior for large $|k|$:

(i) $f(x) = 1 + x$

(ii) $g(x) = \sin 2x$

(iii) $h(x) = \begin{cases} +1 & x \in [0, +\pi] \\ 0 & x \in [-\pi, 0) \end{cases}$

(iv) $j(x) = \begin{cases} +1 & x \in [0, +\pi] \\ -1 & x \in [-\pi, 0) \end{cases}$

35. The Pauli matrices

Consider the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Prove $\sigma_j \, \sigma_k = \delta_{jk} \, \mathrm{id}_{\mathbb{C}^2} + \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \, \sigma_l$ where ϵ_{jkl} is the epsilon tensor.

(ii) Prove that any 2×2 matrix can be written as the linear combination of the identity and the three Pauli matrices with coefficients h_0 and $h = (h_1, h_2, h_3)$,

$$\mathrm{Mat}_{\mathbb{C}}(2) \ni A = (a_{jk})_{1 \leq j,k \leq 2} = h_0 \, \mathrm{id}_{\mathbb{C}^2} + \sum_{j=1}^{3} h_j \, \sigma_j =: \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma. \tag{1}$$

Hint: Use that $\mathrm{Mat}_{\mathbb{C}}(2)$ is finite-dimensional.

(iii) Now assume that the coefficients $h_0, \ldots, h_3$ in equation (1) are real. Show that then the resulting matrix $H = h_0 \, \mathrm{id}_{\mathbb{C}^2} + h \cdot \sigma$ is hermitian. Compute the eigenvalues $E_\pm(h_0, h)$ of H in terms of the coefficients h_0 and h.

(iv) Use (i) to prove that for real $h_0, \ldots, h_3$

$$P_\pm(h_0, h) = \frac{1}{2} \left(\mathrm{id}_{\mathbb{C}^2} \pm \frac{h \cdot \sigma}{|h|} \right), \qquad h \neq 0 \in \mathbb{R}^3, \; |h| := \sqrt{h_1^2 + h_2^2 + h_3^2},$$

are the projections onto the eigenspaces for the two eigenvalues $E_\pm(h_0, h)$ of H.

(v) Compute the trace of H.

Note: In physics especially, one frequently writes $h \cdot \sigma$ for $\sum_{j=1}^{3} h_j \, \sigma_j$ where $h = (h_1, h_2, h_3)$.

36. Functional calculus for 2×2 matrices

Let f be a piecewise continuous function and $H = H^*$ a hermitian 2×2 matrix. Then define

$$f(H) := \sum_{j=\pm} f(E_\pm) \, P_\pm \tag{2}$$

where $E_\pm$ are the eigenvalues of H and $P_\pm$ the two projections from problem 35.

(i) Compute $f(H)$ defined as in equation (2) for $H = h \cdot \sigma$, $h \neq 0$, and

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

(ii) Show that $f(H)$ for $f(x) = \mathrm{e}^{-itx}$ (defined via (2)) coincides with the matrix exponential, i. e.

$$f(H) = \mathrm{e}^{-ith_0} \left(\cos\bigl(|h| \, t\bigr) - \frac{\mathrm{i}}{|h|} \sin\bigl(|h| \, t\bigr) \, h \cdot \sigma \right) = \mathrm{e}^{-\mathrm{i}tH} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!} \, H^n. \tag{3}$$

Hint: Use $\mathrm{e}^{-\mathrm{i}t(h_0 + h \cdot \sigma)} = \mathrm{e}^{-\mathrm{i}th_0} \, \mathrm{e}^{-\mathrm{i}th \cdot \sigma}$.

(iii) Assuming h_0, h_1, h_2, h_3 are real, compute $\psi(t)$ for the initial condition $\psi(0) = \psi_0 \in \mathbb{C}^2$:

(a) $\frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = \bigl(h_2 \, \sigma_2 + h_3 \, \sigma_3\bigr) \psi(t)$

(b) $\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = h_2 \, \sigma_2 \psi(t)$

(c) $-\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = \bigl(h_0 \, \mathrm{id}_{\mathbb{C}^2} + h_3 \, \sigma_3\bigr) \psi(t)$

37. A simple model for graphene

Consider the nearest-neighbor model for graphene

$$H = \begin{pmatrix} q_3 \, \mathrm{id}_{\ell^2(\mathbb{Z}^2)} & 1_{\ell^2(\mathbb{Z}^2)} + q_1 \, \mathfrak{s}_1 + q_2 \, \mathfrak{s}_2 \\ 1_{\ell^2(\mathbb{Z}^2)} + q_1 \, \mathfrak{s}_1^* + q_2 \, \mathfrak{s}_2^* & -q_3 \, \mathrm{id}_{\ell^2(\mathbb{Z}^2)} \end{pmatrix}.$$

Here, $q_1, q_2 \in \mathbb{R}$ are hopping amplitudes while $q_3 \in \mathbb{R}$ is the so-called stagger parameter. Repeat the analysis in Chapter 6.1.5.2:

(i) Compute the momentum representation $H^{\mathcal{F}} := \mathcal{F}^{-1} \, H \, \mathcal{F}$. What Hilbert space does this operator act on?

(ii) Find a matrix-valued function $T(k)$ so that $H^{\mathcal{F}} = T(\hat{k})$ is the multiplication operator associated to T.

(iii) Find the eigenvalues $E_\pm(k)$ and eigenprojections $P_\pm(k)$ of $T(k)$.

(iv) Compute the unitary evolution group $U^{\mathcal{F}}(t)$ for $H^{\mathcal{F}}$.

(v) Compute the unitary evolution group $U(t)$ in position representation.

(vi) **Voluntary:** Identify the parameter region where the eigenvalues are not separated by a gap, i. e. $\inf_{k \in \mathbb{T}^n} \bigl|E_+(k) - E_-(k)\bigr| = 0$.

Remark: This nearest-neighbor model with stagger was used to investigate the piezoelectric effect in graphene: *Topological Polarization in Graphene-like Systems*, G. De Nittis and M. Lein, J. Phys. A **46** no. 38, p. 385001, 2013

Hand in home work on: Thursday, 28 November 2013, before class