The continuous Fourier transform

Homework Problems

38. Unitarity of the Fourier transform

Show that the continuous Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is unitary.

Solution:

We will only show $\mathcal{F}^* \mathcal{F} = \mathrm{id}_{L^2(\mathbb{R}^n)}$, the arguments for $\mathcal{F} \mathcal{F}^* = \mathrm{id}_{L^2(\mathbb{R}^n)}$ are analogous. Parseval's theorem states that

$$\langle \varphi, \psi \rangle = \big\langle \mathcal{F}\varphi, \mathcal{F}\psi \big\rangle = \big\langle \varphi, \mathcal{F}^* \mathcal{F}\psi \big\rangle$$

holds for all $\varphi, \psi \in L^2(\mathbb{R}^n)$. In other words, $\mathcal{F}^* \mathcal{F}\psi - \psi$ is orthogonal to all vectors, and thus it is necessarily 0 which proves $\mathcal{F}^* \mathcal{F} = \mathrm{id}_{L^2(\mathbb{R}^n)}$.

39. Fourier transforms of particular functions (18 points)

Compute the Fourier transforms of the following $L^1(\mathbb{R})$ functions:

(i) $f(x) = \mathrm{e}^{-\lambda|x|}$, $\lambda > 0$

(ii) $g(x) = 1_{[-1,+1]}(x) := \begin{cases} 1 & x \in [-1,+1] \\ 0 & x \notin [-1,+1] \end{cases}$

(iii) $h(x) = x\,\mathrm{e}^{-\frac{\lambda}{2}x^2}$, $\lambda > 0$

(iv) $j = g * g$

(v) $k = \mathrm{e}^{-\frac{\lambda}{2}x^2} * 1_{[-1,+1]}$

Solution:

(i)

$$\begin{aligned}(\mathcal{F}f)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{e}^{-\lambda|x|} \\ &\overset{[1]}{=} \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{e}^{-\lambda x} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{e}^{+\lambda x} \\ &\overset{[1]}{=} \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}x\, \mathrm{e}^{-(\lambda+\mathrm{i}\xi)x} - \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}x\, \mathrm{e}^{-(\lambda-\mathrm{i}\xi)x} \\ &\overset{[1]}{=} \left[\frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{-(\lambda+\mathrm{i}\xi)x}}{-(\lambda+\mathrm{i}\xi)} \right]_0^{+\infty} + \left[\frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{-(\lambda-\mathrm{i}\xi)x}}{-(\lambda-\mathrm{i}\xi)} \right]_0^{+\infty} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\lambda+\mathrm{i}\xi} + \frac{1}{\lambda-\mathrm{i}\xi} \right) \overset{[1]}{=} \sqrt{\frac{2}{\pi}} \frac{\lambda}{\xi^2+\lambda^2}\end{aligned}$$

(ii)

$$\begin{aligned}(\mathcal{F}g)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x\xi}\, 1_{[-1,+1]}(x) \overset{[1]}{=} \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x\xi} \\ &= \left[\frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{-\mathrm{i}x\xi}}{-\mathrm{i}\xi} \right]_{-1}^{+1} \overset{[1]}{=} \sqrt{\frac{2}{\pi}} \frac{\sin\xi}{\xi}\end{aligned}$$

(iii)

$$\begin{aligned}(\mathcal{F}h)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x\xi}\, x\, \mathrm{e}^{-\frac{\lambda}{2}x^2} \\ &\overset{[2]}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x \left(\left(x + \mathrm{i}\tfrac{\xi}{\lambda}\right) - \mathrm{i}\tfrac{\xi}{\lambda} \right) \mathrm{e}^{-\frac{\lambda}{2}(x+\mathrm{i}\frac{\xi}{\lambda})^2}\, \mathrm{e}^{-\frac{1}{2\lambda}\xi^2} \\ &\overset{[2]}{=} \mathrm{e}^{-\frac{1}{2\lambda}\xi^2} \frac{1}{\sqrt{2\pi}} \underbrace{\int_{\mathbb{R}} \mathrm{d}x\, (-\lambda^{-1})\, \partial_x \left(\mathrm{e}^{-\frac{\lambda}{2}(x+\mathrm{i}\frac{\xi}{\lambda})^2} \right)}_{=0} + \frac{\mathrm{i}\xi\, \mathrm{e}^{-\frac{1}{2\lambda}\xi^2}}{\lambda} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{\lambda}{2}(x+\mathrm{i}\frac{\xi}{\lambda})^2}\end{aligned}$$

The integral

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{\lambda}{2}(x+\mathrm{i}\frac{\xi}{\lambda})^2} = \frac{1}{\sqrt{\lambda}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}x^2} \overset{[1]}{=} \frac{1}{\sqrt{\lambda}}$$

has already been computed in the lecture notes (cf. page 101), and thus

$$(\mathcal{F}h)(\xi) \overset{[1]}{=} \frac{\mathrm{i}\xi\, \mathrm{e}^{-\frac{1}{2\lambda}\xi^2}}{\lambda^{3/2}}.$$

(iv)

$$(\mathcal{F}j)(\xi) \overset{[1]}{=} \sqrt{2\pi}\,\big(\mathcal{F}g\big)^2(\xi) \overset{[1]}{=} \sqrt{2\pi}\,\left(-\frac{\mathrm{i}}{\sqrt{2\pi}}\frac{\sin\xi}{\xi}\right)^2 \overset{[1]}{=} -\frac{1}{\sqrt{2\pi}}\frac{\sin^2\xi}{\xi^2}$$

(v)

$$\begin{aligned}(\mathcal{F}k)(\xi) &\overset{[1]}{=} \sqrt{2\pi}\,\big(\mathcal{F}\mathrm{e}^{-\frac{\lambda}{2}x^2}\big)(\xi)\,\big(\mathcal{F}1_{[-1,+1]}\big)(\xi) \\ &\overset{[1]}{=} \sqrt{2\pi}\,\left(\frac{\mathrm{e}^{-\frac{1}{2\lambda}\xi^2}}{\sqrt{\lambda}}\right)\,\left(-\frac{\mathrm{i}}{\sqrt{2\pi}}\frac{\sin\xi}{\xi}\right) \overset{[1]}{=} -\mathrm{i}\,\frac{\sin\xi}{\xi}\,\mathrm{e}^{-\frac{1}{2\lambda}\xi^2}\end{aligned}$$

40. Inhomogeneous heat equation

Consider the inhomogeneous heat equation

$$\partial_t u(t) = +D\,\Delta_x u(t) + f(t), \qquad\qquad u(0) = u_0, \tag{1}$$

with diffusion constant $D > 0$.

(i) Derive the solution $u(t)$ to the inhomogeneous equation. You need not justify your manipulations.

(ii) Verify that the solution from (i) solves (1).

(iii) Give sufficient conditions on the solution u which ensure uniqueness. Justify your answer.

Solution:

(i) In momentum representation, the inhomogeneous heat equation reads

$$\partial_t \hat{u}(t) = -D\,\hat{\xi}^2\,\hat{u}(t) + \widehat{f(t)}, \qquad\qquad \hat{u}(0) = \hat{u}_0,$$

where $\hat{u}(t) = \mathcal{F}\big(u(t)\big)$, $\widehat{f(t)} = \mathcal{F}\big(f(t)\big)$ and $\hat{u}_0 = \mathcal{F}u_0$.

Inhomogeneous linear ODEs have a closed-form solution (equation (2.12) in the lecture notes),

$$\hat{u}(t) = \mathrm{e}^{-tD\hat{\xi}^2}\,\hat{u}_0 + \int_0^t \mathrm{d}s\,\mathrm{e}^{-(t-s)D\hat{\xi}^2}\,\widehat{f(s)},$$

and applying the inverse Fourier transform to $\hat{u}(t)$ yields

$$u(t) = G(t) * u_0 + \int_0^t \mathrm{d}s\,G(t-s) * f(s).$$

(ii) First of all, $u(t)$ satisfies the initial condition,

$$u(0) = \lim_{t \searrow 0}\left(G(t) * u_0 + \int_0^t \mathrm{d}s\,G(t-s) * f(s)\right) = u_0.$$

It is easier to compute the time derivative in the momentum representation:

$$\begin{aligned}\partial_t \hat{u}(t) &= -D\,\hat{\xi}^2\,\mathrm{e}^{-tD\hat{\xi}^2}\,\hat{u}_0 - D\,\hat{\xi}^2 \int_0^t \mathrm{d}s\,\mathrm{e}^{-(t-s)D\hat{\xi}^2}\,\widehat{f(s)} + \mathrm{e}^{-(t-s)D\hat{\xi}^2}\,\widehat{f(s)}\Big|_{s=t} \\ &= -D\,\hat{\xi}^2\,\hat{u}(t) + \widehat{f(t)}\end{aligned}$$

This is just the heat equation in momentum representation, and thus, $u(t)$ solves the initial value problem.

(iii) Just like in the homogeneous case $u(t), \partial_t u(t) \in L^1(\mathbb{R}^n)$ for all $t \geq 0$ is a sufficient condition to ensure uniqueness.

Suppose $\tilde{u}(t)$ were a second solution to the inhomogeneous heat equation, then their difference $g(t) := u(t) - \tilde{u}(t)$ is a solution to the *homogeneous* heat equation and initial value $g(0) = 0$. And the only solution which satisfies $g(t), \partial_t g(t) \in L^1(\mathbb{R}^n)$ is the trivial solution $g(t) = 0$ (Theorem 6.2.15).

41. Uncertainty of Gauß functions

Compute the right-hand side of Heisenberg's uncertainty principle

$$\sigma_\psi(\hat{x})\,\sigma_\psi(-\mathrm{i}\hbar\partial_x)$$

in one dimension for

(i) $\psi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}}\mathrm{e}^{-\frac{\lambda}{2}x^2}$, $\lambda > 0$, and

(ii) $\varphi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}}\mathrm{e}^{+\mathrm{i}x\xi_0}\,\mathrm{e}^{-\frac{\lambda}{2}(x-x_0)^2}$, $\lambda > 0$, $x_0, \xi_0 \in \mathbb{R}$.

Here, the standard deviation

$$\sigma_\psi(H) := \sqrt{\mathbb{E}_\psi\Big(\big(H - \mathbb{E}_\psi(H)\big)^2\Big)}$$

for a selfadjoint operator $H = H^*$ with respect to ψ, $\|\psi\| = 1$, is defined as in the lecture notes via the expectation value

$$\mathbb{E}_\psi(H) := \langle\psi, H\psi\rangle.$$

Solution:

(i) First of all, since $\psi(-x) = \psi(x)$ the expectation value

$$\mathbb{E}_{\psi_\lambda}(\hat{x}) = \int_{\mathbb{R}} \mathrm{d}x\,\sqrt{\frac{\lambda}{\pi}}\,x\,\mathrm{e}^{-\lambda x^2} = 0$$

necessarily vanishes. Similarly,

$$\begin{aligned}\mathbb{E}_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x) &= -\mathrm{i}\hbar\,\big\langle\psi_\lambda, \partial_x\psi_\lambda\big\rangle = -\mathrm{i}\hbar\,\big\langle\mathcal{F}\psi_\lambda, \mathcal{F}\partial_x\psi_\lambda\big\rangle \\ &= \hbar\,\big\langle\psi_{1/\lambda}, \hat{\xi}\psi_{1/\lambda}\big\rangle = 0\end{aligned}$$

is also 0, because the Fourier transform of a Gaußian is also a Gaußian.

That means we can compute the first standard deviation by partial integration:

$$\begin{aligned}\sigma_{\psi_\lambda}(\hat{x})^2 &= \mathbb{E}_{\psi_\lambda}\Big(\big(\hat{x} - \mathbb{E}_{\psi_\lambda}(\hat{x})\big)^2\Big) = \mathbb{E}_{\psi_\lambda}(\hat{x}^2) = \sqrt{\frac{\lambda}{\pi}}\int_{\mathbb{R}}\mathrm{d}x\,x^2\,\mathrm{e}^{-\lambda x^2} \\ &= \frac{1}{\lambda\sqrt{\pi}}\int_{\mathbb{R}}\mathrm{d}x\,x^2\,\mathrm{e}^{-x^2} = \left[-\frac{1}{2\lambda\sqrt{\pi}}\,x\,\mathrm{e}^{-x^2}\right]_{-\infty}^{+\infty} + \frac{1}{2\lambda\sqrt{\pi}}\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{-x^2} = \frac{1}{2\lambda}\end{aligned}$$

To compute the other standard deviation, we note that since the Fourier transform of a Gaußian is a Gaußian with inverse width,

$$(\mathcal{F}\psi_\lambda)(\xi) = \sqrt[4]{\frac{\lambda}{\pi}}\,\big(\mathcal{F}\mathrm{e}^{-\frac{\lambda}{2}x^2}\big)(\xi) = \frac{1}{\sqrt[4]{\lambda\pi}}\mathrm{e}^{-\frac{1}{2\lambda}\xi^2} = \psi_{1/\lambda},$$

we can relate $\sigma_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x)$ to $\sigma_{\psi_\lambda}(\hat{x})$,

$$\begin{aligned}\sigma_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x)^2 &= -\hbar^2\,\big\langle\psi_\lambda, \partial_x^2\psi_\lambda\big\rangle = -\hbar^2\,\big\langle\mathcal{F}\psi_\lambda, \mathcal{F}\partial_x^2\psi_\lambda\big\rangle \\ &= +\hbar^2\,\big\langle\mathcal{F}\psi_{1/\lambda}, \hat{\xi}^2\psi_{1/\lambda}\big\rangle = \frac{\hbar^2}{2\lambda^{-1}}.\end{aligned}$$

Hence, ψ_λ minimizes the uncertainty relation,

$$\sigma_{\psi_\lambda}(\hat{x})\,\sigma_{\psi_\lambda}(-\mathrm{i}\hbar\partial_x) = \frac{1}{\sqrt{2\lambda}}\sqrt{\frac{\hbar^2\lambda}{2}} = \frac{\hbar}{2} \geq \frac{\hbar}{2}.$$

(ii) We will reuse the results from (i) as much as possible: the mean of φ_λ is x_0:

$$\begin{aligned}\mathbb{E}_{\varphi_\lambda}(\hat{x}) &= \int_{\mathbb{R}} \mathrm{d}x \, x \, \Bigl| \mathrm{e}^{+\mathrm{i}x\xi_0} \, \psi_\lambda(x - x_0) \Bigr|^2 = \int_{\mathbb{R}} \mathrm{d}x \, (x + x_0) \, \bigl| \psi_\lambda(x) \bigr|^2 \\ &= x_0 \, \bigl\| \psi_\lambda(x) \bigr\|^2 = x_0\end{aligned}$$

Hence, the standard deviation of φ_λ coincides with that of ψ_λ:

$$\begin{aligned}\sigma_{\varphi_\lambda}(\hat{x})^2 = \mathbb{E}_{\varphi_\lambda}\bigl((\hat{x} - x_0)^2\bigr) &= \int_{\mathbb{R}} \mathrm{d}x \, (x - x_0)^2 \, \Bigl| \mathrm{e}^{+\mathrm{i}x\xi_0} \, \psi_\lambda(x - x_0) \Bigr|^2 \\ &= \int_{\mathbb{R}} \mathrm{d}x \, x^2 \, \bigl| \psi_\lambda(x) \bigr|^2 = \frac{1}{2\lambda}\end{aligned}$$

Since the Fourier transform intertwines taking derivatives with multiplying by monomials and maps Gaußians on Gaußians of inverse width,

$$(\mathcal{F}\varphi_\lambda)(\xi) = \bigl(\mathcal{F}\mathrm{e}^{+\mathrm{i}x\xi_0} \, \psi_\lambda(\,\cdot\, - x_0)\bigr)(\xi) = \bigl(\mathcal{F}\psi_\lambda(\,\cdot\, - x_0)\bigr)(\xi - \xi_0) = \mathrm{e}^{-\mathrm{i}\xi x_0} \, \psi_{1/\lambda}(\xi - \xi_0),$$

we obtain the same integral (up to $\hbar^2$) where λ is replaced by λ^{-1},

$$\sigma_{\varphi_\lambda}(-\mathrm{i}\hbar\partial_x)^2 = \frac{\hbar^2\lambda}{2}.$$

Hence, also shifted Gaußians have minimal uncertainty,

$$\sigma_{\varphi_\lambda}(\hat{x}) \, \sigma_{\varphi_\lambda}(-\mathrm{i}\hbar\partial_x) = \frac{\hbar}{2} \geq \frac{\hbar}{2}.$$