UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics

(APM 421 H)

Winter 2014
Problem Sheet 11
(2014.11.21)

Functional Calculus

Homework Problems

34. The Weyl criterion

Prove the Weyl criterion:

Theorem 1 *Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(H)$.*

(i) $\lambda \in \sigma(H)$ holds if and only if there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ so that $\|\psi_n\| = 1$ and

$$\lim_{n\to\infty} \big\| H\psi_n - \lambda\,\psi_n \big\|_{\mathcal{H}} = 0.$$

(ii) We have $\lambda \in \sigma_{\mathrm{ess}}(H)$ if and only if we can choose the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ to be orthonormal.

35. Functional calculus for the momentum operator

Consider the momentum operator $P = -\mathrm{i}\partial_x$ on $L^2(\mathbb{R})$ with domain $\mathcal{D}(P) = H^1(\mathbb{R})$.

(i) Show $P = P^*$ and give $\sigma(P)$.

(ii) Compute the projection-valued measure $1_\Lambda(P)$ where $\Lambda \subseteq \mathbb{R}$ is a Borel set.

(iii) Explain how to define $U(t) := \mathrm{e}^{-\mathrm{i}tP}$ and prove that $U(t)\,U(s) = U(t+s)$.

(iv) Define the selfadjoint operator $H = P^2$ via the functional calculus associated to P and prove that it coincides with $H' = -\partial_x^2$ endowed with domain $\mathcal{D}(H') = H^2(\mathbb{R})$.

36. Functional calculus for the position operator

Suppose $H = -\partial_x^2 + V = H^*$ is a selfadjoint operator on $L^2(\mathbb{R})$ with domain $\mathcal{D}(H)$, and consider the position operator $Q = \hat{x}$ equipped with domain

$$\mathcal{D}(Q) = \big\{ \varphi \in L^2(\mathbb{R}) \;\big|\; \hat{x}\varphi \in L^2(\mathbb{R}) \big\}.$$

You may use without proof that Q is selfadjoint.

(i) Show that $Q(t) := \mathrm{e}^{+\mathrm{i}tH}\, Q\, \mathrm{e}^{-\mathrm{i}tH}$ satisfies the Heisenberg equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(t) = \mathrm{i}\,\big[H, Q(t)\big].$$

A formal computation suffices (i. e. you may ignore questions of domains).

(ii) Prove that also $Q(t) = Q(t)^*$ is selfadjoint.

(iii) Let $\big(V(\hat{x})\psi\big)(x) := V(x)\,\psi(x)$ be the multiplication operator associated to a bounded Borel function $V : \mathbb{R} \longrightarrow \mathbb{C}$. Prove that $V(\hat{x})$ coincides with $V(Q)$ (defined through functional calculus associated to Q).

(iv) Prove $\big(V(Q)\big)(t) := \mathrm{e}^{+\mathrm{i}tH}\, V(Q)\, \mathrm{e}^{-\mathrm{i}tH}$ coincides with $V\big(Q(t)\big)$.

Hand in home work on: Friday, 28 November 2014, before class