UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 11
(2014.01.09)

The continuous Fourier transform

Homework Problems

38. Unitarity of the Fourier transform

Show that the continuous Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is unitary.

39. Fourier transforms of particular functions

Compute the Fourier transforms of the following $L^1(\mathbb{R})$ functions:

(i) $f(x) = \mathrm{e}^{-\lambda |x|}$, $\lambda > 0$

(ii) $g(x) = 1_{[-1,+1]}(x) := \begin{cases} 1 & x \in [-1,+1] \\ 0 & x \not\in [-1,+1] \end{cases}$

(iii) $h(x) = x \, \mathrm{e}^{-\frac{\lambda}{2} x^2}$, $\lambda > 0$

(iv) $j = g * g$

(v) $k = \mathrm{e}^{-\frac{\lambda}{2} x^2} * 1_{[-1,+1]}$

40. Inhomogeneous heat equation

Consider the inhomogeneous heat equation

$$\partial_t u(t) = +D \, \Delta_x u(t) + f(t), \qquad u(0) = u_0, \tag{1}$$

with diffusion constant $D > 0$.

(i) Derive the solution $u(t)$ to the inhomogeneous equation. You need not justify your manipulations.

(ii) Verify that the solution from (i) solves (1).

(iii) Give sufficient conditions on the solution u which ensure uniqueness. Justify your answer.

41. Uncertainty of Gauß functions

Compute the right-hand side of Heisenberg's uncertainty principle

$$\sigma_\psi(\hat{x}) \, \sigma_\psi(-\mathrm{i}\hbar \partial_x)$$

in one dimension for

(i) $\psi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}} \mathrm{e}^{-\frac{\lambda}{2} x^2}$, $\lambda > 0$, and

(ii) $\varphi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}} \mathrm{e}^{+\mathrm{i} x \xi_0} \, \mathrm{e}^{-\frac{\lambda}{2} (x - x_0)^2}$, $\lambda > 0$, $x_0, \xi_0 \in \mathbb{R}$.

Here, the standard deviation

$$\sigma_\psi(H) := \sqrt{\mathbb{E}_\psi\Big(\big(H - \mathbb{E}_\psi(H)\big)^2\Big)}$$

for a selfadjoint operator $H = H^*$ with respect to ψ, $\|\psi\| = 1$, is defined as in the lecture notes via the expectation value

$$\mathbb{E}_\psi(H) := \langle \psi, H\psi \rangle.$$

Hand in home work on: Thursday, 16 January 2014, before class